PROJECTED WALLPAPER PATTERNS

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Abstract. Consider a periodic function f of two variables with symmetry Γ and let $\mathcal{L} \subset \Gamma$ be the subgroup of translations. The Fourier expansion of a periodic function is a sum over $\mathcal{L}^*$, the dual of the the set $\mathcal{L}$ of all the periods of f. After projecting f, some of its original symmetry remains. We describe the symmetries of the projected function, starting from Γ and from the structure of $\mathcal{L}^*$.

1. Introduction and preliminaries

An usual method of studying bifurcation [5] on problems equivariant under the Euclidean group $\mathbf{E}(2)$ is to look for periodic solutions — see [2, 3, 4]. If $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ has two noncolinear periods then its symmetry group is a plane crystalographic group, $\Gamma \leq \mathbf{E}(2)$, and its level sets form a periodic pattern.

We start with a pattern in $\mathbf{R}^2$ and project it into $\mathbf{R}$. What are the symmetries of the projected pattern? This question is addressed in [6]. The new pattern, the level sets of a function in $\mathbf{R}$, may be periodic or invariant under reflections. We relate the existence of these symmetries to properties of Γ and of $\mathcal{L}^*$, the dual of the set $\mathcal{L}$ of all the periods of f. The set $\mathcal{L}^*$ arises naturally in the Fourier expansion of f and the symmetries in Γ impose restrictions on Fourier coefficients.

We write elements of $\mathbf{E}(2) = \mathbf{R}^2 \dot{+} \mathbf{O}(2)$ in the form (v_δ, δ), whith $v_\delta \in \mathbf{R}^2$ representing a translation and $\delta \in \mathbf{O}(2)$. They act in $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ with the scalar action (see [7]):

$$(v_\delta, \delta) \cdot f(x) = f((v_\delta, \delta)^{-1}) \cdot x) = f(\delta^{-1} x - \delta^{-1} v_\delta).$$

We assume that Γ is a plane crystalographic group — see [1, 8] for general results and definitions. Denote by $\mathcal{L}$ the subgroup of the translations in Γ, a module over the integers, also called a lattice. If f is Γ-invariant, then in particular elements of $\mathcal{L}$ are periods of f. A pattern and the lattice $\mathcal{L}$ may not have the same symmetries: see figure 1.

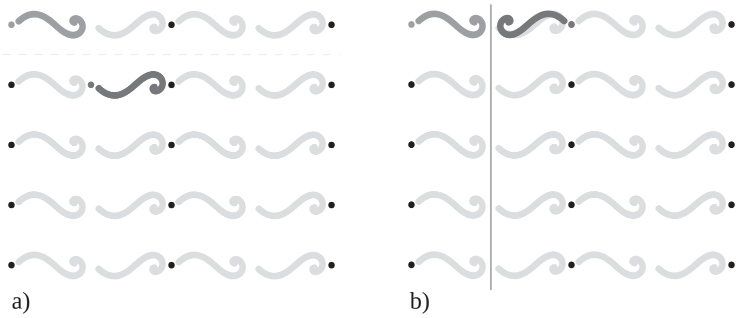

FIGURE 1. a) The lattice (black dots) is not invariant under the glide reflection transforming the grey motive into the darker one. However this is a symmetry of the lighter pattern. b) The lighter pattern is not invariant under the reflection on the black line, although this is a symmetry of the lattice (black dots).

2. Symmetries and Projection

Let X_Γ be a vector space of Γ-invariant functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, having unique formal Fourier expansions of the form:

$$f(x,y) = \sum_{k \in \mathcal{L}^*} \omega_k(x,y) C(k),$$

where $\mathcal{L}^*$ is the dual lattice and $\omega_k(x,y) = \mathrm{e}^{2\pi i < k,(x,y)>}$.

The elements of $\mathcal{L}^*$ are $k \in \mathbf{R}^2$ such that $< k, l > \in \mathbf{Z}$ for all $l \in \mathcal{L}$, where $< k, l >$ is the usual inner product in $\mathbf{R}^2$.

Given $y_0 > 0$, define the projection of a function $f \in X_\Gamma$ to be the function

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy \qquad x, y \in \mathbf{R}.$$

We assume that in X_Γ we have,

$$\Pi_{y_0}(f)(x) = \sum_{k \in \mathcal{L}^*} \int_0^{y_0} \omega_k(x,y) C(k) dy$$

and that X_Γ contains, for all $k \in \mathcal{L}^*$, the real and imaginary parts of $I_k(x,y) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k}(-v_\delta)\omega_{\delta k}(x,y)$, where $\mathbf{J} \sim \Gamma/\mathcal{L}$ is the largest subgroup of $\mathbf{O}(2)$ that leaves $\mathcal{L}$ invariant. Notice that these are the simplest Γ-invariant functions.

The first step in obtaining the symmetries of the projected functions is to relate the (v_α, α)-invariance to restrictions on Γ and on $\mathcal{L}^*$. This is the main result in this paper: Proposition 2.1, below.

For $\alpha \in \{1, -1\}$, let $\alpha_+ \in \{I, -\sigma\}$ and $\alpha_- = \sigma\alpha_+ \in \{\sigma, -I\}$, where

$$\alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that $\alpha_\pm = \alpha_\pm^{-1}$ and $\sigma = \sigma^{-1}$.

Proposition 2.1. *All functions in* $\Pi_{y_0}(X_\Gamma)$ *are invariant under the action of* $(v_\alpha, \alpha) \in \mathbf{R}\dot{+}\mathbf{O}(1)$ *if and only if one of the following conditions holds:*

A. $(v_+, \alpha_+) \in \Gamma$ *and for each* $k \in \mathcal{L}^*$,
either $< k, (0, y_0) >\in \mathbf{Z} - \{0\}$ *or* $< k, v_+ - (v_\alpha, 0) >\in \mathbf{Z}$,

B. $(v_-, \alpha_-) \in \Gamma$ *and for each* $k \in \mathcal{L}^*$,
either $< k, (0, y_0) >\in \mathbf{Z} - \{0\}$ *or* $< k, v_- - (v_\alpha, y_0) >\in \mathbf{Z}$,

C. (v_σ, σ), $(v_+, \alpha_+) \in \Gamma$ *and, for each* $k \in \mathcal{L}^*$, *one of the conditions C1, C2 or C3 below holds:*

C1. $< k, (0, y_0) >\in \mathbf{Z} - \{0\}$,
C2. $< k, v_+ - (v_\alpha, 0) >\in \mathbf{Z}$,
C3. $< k, v_\sigma - (0, y_0) > +\frac{1}{2} \in \mathbf{Z}$.

A more concise formulation of this result is possible using the subsets of $\mathcal{L}^*$ defined below. Let $\mathcal{M}^*_+$ and $\mathcal{M}^*_-$ be the modules

$$\mathcal{M}^*_+ = \{k \in \mathcal{L}^* : < k, v_+ - (v_\alpha, 0) >\in \mathbf{Z}\} \text{ and}$$
$$\mathcal{M}^*_- = \{k \in \mathcal{L}^* : < k, v_- - (v_\alpha, y_0) >\in \mathbf{Z}\},$$

and let

$$\mathcal{N}^*_{y_0} = \{k \in \mathcal{L}^* : < k, (0, y_0) >\in \mathbf{Z} - \{0\}\},$$
$$\mathcal{N}^*_\sigma = \{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) > +1/2 \in \mathbf{Z}\}.$$

The last two sets are not modules. The smallest modules generated by each of them are, respectively, $\overline{\mathcal{N}^*_{y_0}} = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{y_0}$ and $\overline{\mathcal{N}^*_\sigma} = \mathcal{N}^*_\sigma \cup \mathcal{M}^*_\sigma$, where all the unions are disjoint and $\mathcal{M}^*_{y_0}$ and $\mathcal{M}^*_\sigma$ are the modules

$$\mathcal{M}^*_{y_0} = \{k \in \mathcal{L}^* : < k, (0, y_0) >= 0\} \text{ and}$$
$$\mathcal{M}^*_\sigma = \{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) >\in \mathbf{Z}\}.$$

Properties of $\mathcal{N}^*_\sigma$: Let $m_1, m_2 \in \mathbf{Z}$. If $g_1, g_2 \in \mathcal{N}^*_\sigma$ then

$$m_1 g_1 + m_2 g_2 \in \begin{cases} \mathcal{M}^*_\sigma & \text{if } m_1 + m_2 \text{ even} \\ \mathcal{N}^*_\sigma & \text{if } m_1 + m_2 \text{ odd} \end{cases}. \tag{1}$$

Proposition 2.1 can therefore be written the following way:

Proposition 2.2. *All functions in* $\Pi_{y_0}(X_\Gamma)$ *are invariant under the action of* $(v_\alpha, \alpha) \in \mathbf{R}\dot{+}\mathbf{O}(1)$ *if and only if one of the following conditions holds:*

A. $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$,
B. $(v_-, \alpha_-) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$,
C. (v_σ, σ), $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma$.

For $D(k_1) = \sum_{k_2 : (k_1, k_2) \in \mathcal{L}^*} C(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y) dy$, the projection of $f \in X_\Gamma$ may be written, with $\mathcal{L}^*_1 = \{k_1 : (k_1, k_2) \in \mathcal{L}^*\}$, as

$$\Pi_{y_0}(f)(x) = \sum_{k_1 \in \mathcal{L}^*_1} \omega_{k_1}(x) D(k_1).$$

Thus $\Pi_{y_0}(f)$ is (v_α, α)-invariant if and only if

$$\sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(\alpha\, x) \omega_{k_1}(-\alpha\, v_\alpha) D(k_1), \tag{2}$$

or, equivalently, $D(k_1) = \omega_{k_1}(-v_\alpha) D(\alpha\, k_1)$, for all $k_1 \in \mathcal{L}_1^*$.

In the next section we show that each condition of Proposition 2.1 leads to the restrictions on the coefficients $D(k_1)$ above. Reciprocally, when those restrictions are imposed on the projection of $I_k(x, y)$, for all $k \in \mathcal{L}^*$, this implies the conditions of Proposition 2.1.

3. Proof of Proposition 2.2

Let $f \in X_\Gamma$ and $(v_\alpha, \alpha) \in \mathbf{R} \dot{+} \mathbf{O}(1)$. If $\Pi_{y_0}(f)$ is (v_α, α)-invariant then $\Pi_{y_0}(f)(x) = \Pi_{y_0}(f)(\alpha\, x - \alpha\, v_\alpha)$, which is equivalent to (2). The right hand side of (2) equals $\sum_{k_1 \in \mathcal{L}_1^*} \omega_{\alpha k_1}(x) \omega_{\alpha k_1}(v_\alpha) D(k_1)$. Since $\alpha(\mathcal{L}_1^*) = (\mathcal{L}_1^*)$ and Fourier expansions are unique, then for each $k_1 \in \mathcal{L}_1^*$, we have:

$$D(k_1) - \omega_{k_1}(-v_\alpha) D(\alpha\, k_1) = 0. \tag{3}$$

Proof — sufficiency. The difference in (3) may be written as

$$\sum_{k_2 : (k_1, k_2) \in \mathcal{L}^*} C(k_1, k_2) G(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y) dy. \tag{4}$$

In each case we compute $G(k_1, k_2)$ and use the conditions on $\mathcal{L}^*$.

Suppose $\alpha_+ \in \mathbf{J}$. Then all the Fourier coefficients of any $f \in X_\Gamma$ satisfy $C(k) = \omega_k(-v_+) C(\alpha\, k)$ and $G(k_1, k_2) = 1 - \omega_k(v_+ - (v_\alpha, 0))$. Thus $G(k_1, k_2) = 0$ if $< k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$.

If $(v_-, \alpha_-) \in \Gamma$ then $G(k_1, k_2) = 1 - \omega_k(v_- - (v_\alpha, y_0))$, since

$$\int_0^{y_0} \omega_{-k_2}(y) dy = \omega_{k_2}(-y_0) \int_0^{y_0} \omega_{k_2}(y) dy. \tag{5}$$

Then $G(k_1, k_2) = 0$ if $< k, v_- - (v_\alpha, y_0) > \in \mathbf{Z}$.

When both (v_+, α_+) and (v_-, α_-) lie in Γ then

$$G(k_1, k_2) = 1 + \omega_k(v_\sigma) \omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha) \left(\omega_k(v_+) + \omega_k(v_-) \omega_{k_2}(-y_0) \right).$$

Using $\omega_k(v_-) = \omega_k(v_\sigma) \omega_k(\sigma v_+)$ and $\omega_k(\sigma v_+ - v_+) = 1$ we get

$$G(k_1, k_2) = \left(1 - \omega_k(v_+ - (v_\alpha, 0))\right) \left(1 + \omega_k(v_\sigma - (0, y_0))\right).$$

If either $1 - \omega_k(v_+ - (v_\alpha, 0)) = 0$ or $1 + \omega_k(v_\sigma - (0, y_0)) = 0$ then $G(k_1, k_2) = 0$.

It follows from the conditions on $\mathcal{L}^*$ that for each $k \in \mathcal{L}^*$ either $\int_0^{y_0} \omega_{k_2}(y) dy = 0$ or $G(k_1, k_2) = 0$ and thus (3) holds for all $k \in \mathcal{L}^*$. $\square$

Proof — necessity. For $D'(\delta, k) = \omega_{\delta k}(-v_\delta) \int_0^{y_0} \omega_{\delta k|_2}(y) dy$, the projections of I_k, with $k \in \mathcal{L}^*$, are

$$\Pi_{y_0}(I_k)(x) = \sum_{\tilde{k}_1 \in \mathbf{J}k|_1} \omega_{\tilde{k}_1}(x) \sum_{\tilde{k}_2 : (\tilde{k}_1, \tilde{k}_2) \in \mathbf{J}k} D'(\delta, \tilde{k}),$$

where $\delta k|_j$ denotes the $j^{\underline{\text{th}}}$ coordinate of δk. If $\Pi_{y_0}(I_k)$ is (v_α, α)-invariant then, by (3),

$$\sum_{\delta \in \mathrm{J}^I(k)} D'(\delta, k) - \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^\alpha(k)} D'(\delta, k) = 0,$$

where $\mathrm{J}^I(k) = \{\delta \in \mathbf{J} : \delta k|_1 = k_1\}$ and $\mathrm{J}^\alpha(k) = \{\delta \in \mathbf{J} : \delta k|_1 = \alpha\, k_1\}$. Let $\mathbf{J}^I = \{I, \sigma\} \cap \mathbf{J}$ and $\mathbf{J}^\alpha = \{\alpha_+, \alpha_-\} \cap \mathbf{J}$. We list some properties of $\mathrm{J}^I(k)$ and $\mathrm{J}^\alpha(k)$ in Lemma 3.1 below. Then we describe the set $\mathcal{O}^* = \left\{k \in \mathcal{L}^* : \mathrm{J}^I(k) = \mathbf{J}^I \wedge \mathrm{J}^\alpha(k) = \mathbf{J}^\alpha\right\}$ in Lemma 3.2. A geometrical characterization of the complement of $\mathcal{O}^*$ in $\mathcal{L}^*$ is given in Lemma 3.3 and in Lemma 3.4 we reformulate the cases of Lemma 3.2 in terms of $\mathcal{L}^*$ instead of $\mathcal{O}^*$, completing the proof. □

Lemma 3.1. *For $k \in \mathcal{L}^*$, the sets $\mathrm{J}^I(k)$ and $\mathrm{J}^\alpha(k)$ satisfy:*

1. $\mathrm{J}^I(k) = \{\delta \in \mathbf{J} : \delta k = k \vee \delta k = \sigma k\}$.
2. $\mathrm{J}^\alpha(k) = \{\delta \in \mathbf{J} : \delta k = \alpha_+ k \vee \delta k = \alpha_- k\}$.
3. $\mathbf{J}^I \subset \mathrm{J}^I(k)$, $\mathbf{J}^\alpha \subset \mathrm{J}^\alpha(k)$ *and* $\mathrm{J}^I(0,0) = \mathrm{J}^\alpha(0,0) = \mathbf{J}$.
4. *Let $k = (k_1, k_2) \neq (0,0)$. If $\delta \in \mathrm{J}^I(k) - \mathbf{J}^I$ then $\delta k = (k_1, -|\delta| k_2)$ and if $\delta \in \mathrm{J}^\alpha(k) - \mathbf{J}^\alpha$ then $\delta k = \alpha(k_1, -|\delta| k_2)$, where $|.|$ is the determinant.*

Proof. Properties 1. and 2. follow by orthogonality of $\mathbf{J}$ and Property 3. is imediate from this and the definitions.

For property 4, let $\delta \in \mathrm{J}^I(k) - \mathbf{J}^I$ and $k \neq (0,0)$. If $\delta k = k$ then $|\delta| = -1$, since an element of $\mathbf{O}(2)$ with determinant 1, other than the identity, does not fix any point besides the origin. Similarly if $\delta k = \sigma k$ then $|\sigma\delta| = -1$ and $|\delta| = 1$. Now suppose $\delta \in \mathrm{J}^\alpha(k) - \mathbf{J}^\alpha$ and $k \neq (0,0)$. Thus, either $\alpha_+\delta = k$ or $\alpha_+\delta = \sigma k$. As $\alpha_+\delta \in \mathrm{J}^I(k) - \mathbf{J}^I$, we may apply the previous result to $\alpha_+\delta$, and the property follows. □

Lemma 3.2. *Suppose that $\sum_{\delta \in \mathrm{J}^I(k)} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^\alpha(k)} D'(\delta, k)$ for all $k = (k_1, k_2) \in \mathcal{L}^*$. Then one of the following cases holds:*

1. $\mathbf{J}^I = \{I\}$, $\mathbf{J}^\alpha = \emptyset$ *and* $\mathcal{O}^* \subset \mathcal{N}^*_{y_0}$,
2. $\mathbf{J}^I = \{I, \sigma\}$, $\mathbf{J}^\alpha = \emptyset$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma\right)$,
3. $\mathbf{J}^I = \{I\}$, $\mathbf{J}^\alpha = \{\alpha_+\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$,
4. $\mathbf{J}^I = \{I\}$, $\mathbf{J}^\alpha = \{\alpha_-\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-\right)$,
5. $\mathbf{J}^I = \{I, \sigma\}$, $\mathbf{J}^\alpha = \{\alpha_+, \alpha_-\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$.

Proof. If $\mathbf{J}^\alpha = \emptyset$ and $k \in \mathcal{O}^*$ then by hypothesis $\sum_{\delta \in \mathbf{J}^I} D'(\delta, k) = 0$. By (5), if $\sigma \in \mathbf{J}$ then $(1 + \omega_k(v_\sigma - (0, y_0))) \int_0^{y_0} \omega_{k_2}(y)dy = 0$ and $\int_0^{y_0} \omega_{k_2}(y)dy = 0$ if $\sigma \notin \mathbf{J}$. Cases 1 and 2 follow because $\int_0^{y_0} \omega_{k_2}(y)dy = 0$ implies $k \in \mathcal{N}^*_{y_0}$ and $1 + \omega_k(v_\sigma - (0, y_0)) = 0$ implies $k \in \mathcal{N}^*_\sigma$.

In case 3 we have $(1 - \omega_{k_1}(-v_\alpha)\omega_k(v_+)) \int_0^{y_0} \omega_{k_2}(y)dy = 0$ and the result follows because $1 - \omega_{k_1}(-v_\alpha)\omega_k(v_+) = 0$ implies $k \in \mathcal{M}^*_+$.

In case 4, $(1 - \omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_{k_2}(-y_0)) \int_0^{y_0} \omega_{k_2}(y)dy = 0$ and either $k \in \mathcal{N}^*_{y_0}$ or $1 - \omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_{k_2}(-y_0) = 0$, which implies $k \in \mathcal{M}^*_-$.

The hypothesis in case 5 yelds $G(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy=0$, where

$$G(k_1,k_2)=1+\omega_k(v_\sigma)\omega_{k_2}(-y_0)-\omega_{k_1}(-v_\alpha)\left(\omega_k(v_+)+\omega_k(v_-)\omega_{k_2}(-y_0)\right),$$

as in the proof of sufficiency in Proposition 2.1. Therefore, either $k\in\mathcal{N}^*_{y_0}$ or $G(k_1,k_2)=0$. In the second case either $(1-\omega_k(v_+-(v_\alpha,0)))=0$ or $(1+\omega_k(v_\sigma-(0,y_0)))=0$ and the result follows. □

Let $\mathcal{P}^*=\left\{k\in\mathcal{L}^*:\ \mathrm{J}^I(k)\neq\mathbf{J}^I\ \vee\ \mathrm{J}^\alpha(k)\neq\mathbf{J}^\alpha\right\}$ be the complement of $\mathcal{O}^*$ in $\mathcal{L}^*$.

Lemma 3.3. *$\mathcal{P}^*$ lies in a finite union of lines through the origin.*

Proof. $\mathcal{P}^*$ may be written as a finite union of submodules

$$\mathcal{P}^*=\bigcup_{\delta\in\mathbf{J}-\mathbf{J}^I}\mathcal{M}^*_{\delta,I}\cup\bigcup_{\delta\in\mathbf{J}-\mathbf{J}^\alpha}\mathcal{M}^*_{\delta,\alpha}$$

for $\mathcal{M}^*_{\delta,\xi}=\{k\in\mathcal{L}^*:\delta k=\xi(k_1,-|\delta|k_2)\}$ and $\xi=I,\alpha$. If δ is a rotation then for $k\in\mathcal{M}^*_{\delta,\xi}$ we have $\delta k=\pm(k_1,-k_2)$, *i.e.*, k lies on the line fixed by $\pm\sigma\delta$. Therefore $\mathcal{M}^*_{\delta,\xi}$ is the intersection of those lines with $\mathcal{L}^*$. Similarly, if δ is a reflection then $\mathcal{M}^*_{\delta,\xi}$ is the intersection of $\mathcal{L}^*$ with a line fixed either by δ or by $-\delta$. □

Lemma 3.4. *If $\sum_{\delta\in\mathrm{J}^I(k)}D'(\delta,k)=\omega_{k_1}(-v_\alpha)\sum_{\delta\in\mathrm{J}^\alpha(k)}D'(\delta,k)$ for all $k=(k_1,k_2)\in\mathcal{L}^*$, then one of the following cases holds:*
A. $\mathbf{J}^\alpha=\{\alpha_+\}$ and $\mathcal{L}^*=\mathcal{N}^*_{y_0}\cup\mathcal{M}^*_+$,
B. $\mathbf{J}^\alpha=\{\alpha_-\}$ and $\mathcal{L}^*=\mathcal{N}^*_{y_0}\cup\mathcal{M}^*_-$,
C. $\mathbf{J}^\alpha=\{\alpha_+,\alpha_-\}$ and $\mathcal{L}^*=\mathcal{N}^*_{y_0}\cup\mathcal{M}^*_+\cup\mathcal{N}^*_\sigma$.

Proof. Let $k\in\mathcal{L}^*-\{(0,0)\}$ and observe that

$$\left(\mathcal{M}^*_{y_0}\cap\mathcal{P}^*\right)-\{(0,0)\}=\emptyset. \tag{6}$$

Let $g=(1/n)k\in\mathcal{L}^*$, $n\in\mathbf{Z}$, have minimal norm and choose $h\in\mathcal{L}^*$ such that $\mathcal{L}^*=\{g,h\}_{\mathbf{Z}}$. Let $\mathcal{Q}^*_k=\{k+mh:m\in\mathbf{Z}\}$. Since $\mathcal{Q}^*_k$ is contained in a line in $\mathbf{R}^2$ that does not go through the origin, by Lemma 3.3, the set $\mathcal{Q}^*_k\cap\mathcal{P}^*$ is finite.

For $k\in\mathcal{L}^*-\{(0,0)\}$ there are three possibilities for $\mathcal{Q}^*_k\cap\overline{\mathcal{N}^*_{y_0}}$: it is either the empty set, or a set with only a point, or an infinite set of equally spaced points. This happens because $\overline{\mathcal{N}^*_{y_0}}$ is a module and if $k+m_1h\neq k+m_2h\in\mathcal{Q}^*_k\cap\overline{\mathcal{N}^*_{y_0}}$, then $(m_2-m_1)h\in\overline{\mathcal{N}^*_{y_0}}$ and $\{k+m_1h+m(m_2-m_1)h:m\in\mathbf{Z}\}$ is a subset of $\left(\mathcal{Q}^*_k\cap\overline{\mathcal{N}^*_{y_0}}\right)$. A characteristic period, τ_{y_0}, is given by the smallest difference between two elements of $\mathcal{Q}^*_k\cap\overline{\mathcal{N}^*_{y_0}}$.

The same three possibilities hold for $\mathcal{Q}^*_k\cap\mathcal{N}^*_\sigma$. Although $\mathcal{N}^*_\sigma$ is not a module, the smallest difference between two elements of $\mathcal{Q}^*_k\cap\mathcal{N}^*_\sigma$ defines a period $\tau_\sigma\in\mathcal{M}^*_\sigma$, by (1). Thus, whenever $\mathcal{Q}^*_k\cap\mathcal{N}^*_\sigma$ has more than one element, if $k+m_1h\in\mathcal{N}^*_\sigma$ then $\{k+m_1h+m\tau_\sigma:m\in\mathbf{Z}\}=\mathcal{Q}^*_k\cap\mathcal{N}^*_\sigma$.

Repeating the construction for $\mathcal{Q}^*_k\cap\mathcal{M}^*_+$ and $\mathcal{Q}^*_k\cap\mathcal{M}^*_-$ we may define characteristic periods τ_+ and τ_-, respectivelly, when these sets have more than one element.

We complete the proof following the cases of Lemma 3.2.

Case 1). From $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{P}^*$, we get $\mathcal{M}_{y_0}^* \subset \mathcal{P}^*$ and, by (6), $\mathcal{M}_{y_0}^* = \{(0,0)\}$. Moreover, $\mathcal{Q}_k^* \cap \mathcal{N}_{y_0}^*$ must be infinite because $\mathcal{Q}_k^* \cap \mathcal{P}^*$ is finite. Thus, the period τ_{y_0} exists and $\mathcal{Q}_k^* - \overline{\mathcal{N}_{y_0}^*}$ is either empty or infinite. From $\left(\mathcal{Q}_k^* - \overline{\mathcal{N}_{y_0}^*}\right) \subset (\mathcal{Q}_k^* \cap \mathcal{P}^*)$ it follows that $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$. Since $\sigma \in \mathbf{J}$, then $\mathcal{M}_{y_0}^* \neq \{(0,0)\}$ and so case 1) cannot occur.

Case 2). Here $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^* \cup \mathcal{P}^*$ which, by (6), implies $\mathcal{M}_{y_0}^* \subset (\mathcal{N}_\sigma^* \cup \{(0,0)\})$. Moreover, $\mathcal{M}_{y_0}^* \neq \{(0,0)\}$ since $\sigma \in \mathbf{J}$. Suppose $\tilde{k} \in \mathcal{M}_{y_0}^*$ and $\tilde{k} \neq (0,0)$, then, $\tilde{k} \in \mathcal{N}_\sigma^*$ and $2\tilde{k} \in \mathcal{M}_{y_0}^*$. However, $2\tilde{k} \notin \mathcal{N}_\sigma^*$, by (1), and so case 2) is also impossible.

Case 3). We follow the arguments of case 1) using the least common multiple of the existing periods, τ_{y_0} or τ_+, instead of τ_{y_0}. Therefore $k \in \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_+^*\right)$ and condition A follows because $(0,0) \in \mathcal{M}_+^*$.

Case 4). This is like case 3) with $\mathcal{M}_-^*$ and τ_- instead of $\mathcal{M}_+^*$ and τ_+, yelding condition B.

Case 5). Here $\mathcal{Q}_k^* - \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_+^* \cup \mathcal{N}_\sigma^*\right) = \emptyset$ because at least one of the periods τ_{y_0}, τ_+ or τ_σ exists and condition C follows. □

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