

ENTROPY FORMULA AND CONTINUITY OF ENTROPY FOR PIECEWISE EXPANDING MAPS

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ABSTRACT. We consider some classes of piecewise expanding maps in finite dimensional spaces having invariant probability measures which are absolutely continuous with respect to Lebesgue measure. We derive an entropy formula for such measures and, using this entropy formula, we present sufficient conditions for the continuity of that entropy with respect to the parameter in some parametrized families of maps. We apply our results to a classical one-dimensional family of tent maps and a family of two-dimensional maps which arises as the limit of return maps when a homoclinic tangency is unfolded by a family of three dimensional diffeomorphisms.

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2010 *Mathematics Subject Classification.* 37A05, 37A10, 37A35, 37C75.

Key words and phrases. Piecewise expanding maps, Metric entropy, Entropy formula.

JFA is partially supported by CMUP (UID/MAT/00144/2013), PTDC/MAT-CAL/3884/2014 and FAPESP/19805/2014, which are funded by FCT (Portugal) with national (MEC) and European structural funds through the programs COMPTE and FEDER, under the partnership agreement PT2020. AP is partially supported by projects MINECO-15-MTM2014-56953-P and MTM2017-87697-P.

1. INTRODUCTION

Be it good or bad, in the mathematical theory of Dynamical Systems one can easily find many examples of systems with simple evolution laws whose dynamics is very complex and hard to predict in deterministic terms. Just to mention a few, in this direction we refer the one-dimensional quadratic maps, the two-dimensional Hénon quadratic diffeomorphisms, or the system of Lorenz quadratic differential equations in the three-dimensional Euclidean space. Though simple in formulation, all these systems have very complicate dynamical behavior and, in the last decades, have motivated the appearance of relevant mathematical results in several directions.

Among the many important contributions to the theory, there are several results both on the existence of *Sinai-Ruelle-Bowen (SRB) measures*, i.e. ergodic invariant probability measures whose conditionals on local unstable manifolds are absolutely continuous with respect to the conditionals of Lebesgue measure, and on the continuous dependence of these SRB measures (or their entropies) with respect to the underlying dynamics; see [1, 4, 5, 6, 8, 9, 11, 12, 13, 14, 19]. In all these situations, the SRB measures are known to be physically relevant, in the sense that they describe the statistics of many initial states of the systems, frequently almost all initial states with respect to the Lebesgue (volume) measure on the ambient space.

Mostly motivated by the family of two-dimensional tent maps introduced in [31] (see Subsection 1.5 below), in this work we present some general results on the continuity of the entropy of ergodic absolutely continuous invariant probability measures for some classes of piecewise expanding maps in any finite dimension, possibly with infinitely many domains of smoothness. As a main application of these results we shall consider the family of two-dimensional tent maps considered in [31]. This family is particularly interesting because it is related to the limit return maps arising when a homoclinic tangency is unfolded by a family of three dimensional diffeomorphisms; see [31] and [36] for details. After having proved in [32] the existence of ergodic absolutely continuous invariant probability measures for these tents maps, and in [7] the continuity of such measures, it is then natural to ask under which conditions the metric entropy with respect to those measures depends continuously on the dynamics.

Our strategy to prove the continuity of the metric entropy is heavily based on the validity of an *entropy formula* for invariant measures, particularly when the measure is absolutely continuous with respect to the reference Lebesgue measure. For the case of smooth diffeomorphisms of a Riemannian manifold, Ruelle established in [35] that the entropy of any invariant probability measure is bounded by the integral of the sum of the positive Lyapunov exponents (counted with multiplicity) with respect to that measure. The reverse inequality has been obtained in [30] by Pesin for the case that the invariant probability measure is absolutely continuous with respect to the Lebesgue measure. Natural versions for non-invertible smooth maps have been drawn in [34]. Extensions of the results of Ruelle and Pesin for the the class of maps with infinite derivative introduced in [23] were obtained in [28]. Conversely, the existence of an entropy formula for can as well be used to prove that an invariant probability measure has conditional measures on local unstable manifolds which are absolutely continuous with respect to the conditionals of Lebesgue measure on those manifolds; see [26, 27, 29].

In the context of non-invertible maps, we are naturally lead to consider the case where all Lyapunov exponents are positive and the sum of Lyapunov exponents coincides with the Jacobian of the map. Surprisingly, we did not find in the literature any result that could be directly used to assure that the tent maps in [31] satisfy the entropy formula. Actually, to the best of our knowledge, not much is known on the existence of entropy formulas for piecewise smooth maps, specially in dimension greater than one. For one-dimensional dynamical systems see e.g. [10], [18], [24] or [26]. In higher dimensions, [17] is the closest to our setting that we could find in the literature. However, a technical assumption in [17, Theorem 1] that we could not be verify in our tent maps, prevented us from applying that result; see condition (2) below. Let us refer that in the Markovian case of piecewise expanding maps with full branches (which is not the case of our tent maps), the situation is completely different: not only the assumptions of [17, Theorem 1] can be easily verified, but also a direct approach as in [6, Section 4] can be implemented to obtain an entropy formula.

Let us point out that, though our initial motivation for this work is the aforementioned two-dimensional family of tent maps, our results on the existence of an entropy formula and continuity of the entropy hold for much more general families of piecewise expanding maps with infinitely many domains of smoothness in any finite dimension.

1.1. Piecewise expansion and bounded distortion. Let Ω be a compact subset of \mathbb{R}^d , for some $d \geq 1$. Consider m the Lebesgue (or volume) measure on Ω and, for each $1 \leq p \leq \infty$, the space $L^p(m)$ endowed with its usual norm $\| \cdot \|_p$. Throughout this paper, absolute continuity will be always meant with respect to the Lebesgue measure m .

Assume that $\phi : \Omega \rightarrow \Omega$ is a map for which there is a (Lebesgue mod 0) countable partition \mathcal{R}_ϕ of Ω such that each $R \in \mathcal{R}_\phi$ is a closed domain with piecewise C^1 boundary of finite $(d - 1)$ -dimensional measure. Assume also that $\phi_R := \phi|_R$ is a C^1 bijection from $\text{int}(R)$, the interior of R , onto its image, with a C^1 extension to R . We say that ϕ is a *piecewise expanding map* if

(P₁) there is $0 < \sigma < 1$ such that for all $R \in \mathcal{R}_\phi$ and all $x \in \text{int}(\phi(R))$

$$\|D\phi_R^{-1}(x)\| \leq \sigma.$$

Consider the Jacobian function

$$J_\phi = |\det(D\phi)|,$$

naturally defined on the (full Lebesgue measure) subset of points in Ω where ϕ is differentiable. We say that ϕ has *bounded distortion* if

(P₂) there is $\Delta \geq 0$ such that for all $R \in \mathcal{R}$ and all $x, y \in \text{int}(R)$

$$\log \frac{J_\phi(x)}{J_\phi(y)} \leq \Delta \|\phi(x) - \phi(y)\|.$$

Sufficient conditions for the existence of an absolutely continuous invariant probability measure for a piecewise expanding map with bounded distortion are given in [21] for finitely many domains of smoothness, and in [1, Section 5] for infinitely many domains. In the latter case, these conditions correspond to property (P₃) and (*) below on the images of the smoothness domains. Similar conditions are imposed on the domains themselves, in the finitely many domains case considered in [21].

1.2. Entropy formula. One of the main goals of this work is to obtain an entropy formula for an absolutely continuous invariant probability measure of a C^1 piecewise expanding map $\phi : \Omega \rightarrow \Omega$. Actually, one of the inequalities will be obtained for invariant measures not necessarily absolutely continuous.

We start by recalling the notion of entropy of ϕ with respect to an invariant measure μ . The *entropy of a partition* \mathcal{P} of Ω with respect to a measure μ is given by

$$H_\mu(\mathcal{P}) = - \sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),$$

the *entropy of ϕ with respect to μ and a partition \mathcal{P}* is given by

$$h_\mu(\phi, \mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\mathcal{P}^n),$$

where for each n

$$\mathcal{P}^n = \bigvee_{j=0}^{n-1} \phi^{-j}(\mathcal{P}). \quad (1)$$

Finally, the *entropy of ϕ with respect to μ* is given by

$$h_\mu(\phi) = \sup_{\mathcal{P}} h_\mu(\phi, \mathcal{P}).$$

We say that the partition \mathcal{R}_ϕ into the smoothness domains of $\phi : \Omega \rightarrow \Omega$ is *quasi-Markovian* with respect to a measure μ , if there exists $\eta > 0$ such that for μ almost every $x \in \Omega$, there are infinitely many values of $n \in \mathbb{N}$ for which

$$m(\phi^n(R^n(x))) \geq \eta,$$

where $R^n(x)$ stands for the element in \mathcal{R}_ϕ^n containing $x \in \Omega$. In this definition we are implicitly assuming that \mathcal{R}_ϕ^n is a μ mod 0 partition of Ω , for all $n \geq 1$.

Our first result relates the entropy with the integral of the Jacobian for a piecewise expanding map, and it holds for invariant probability measures in general, not necessarily absolutely continuous.

Theorem A. *Let ϕ be a C^1 piecewise expanding map with bounded distortion. If μ is a ϕ -invariant probability measure such that $H_\mu(\mathcal{R}_\phi) < \infty$ and \mathcal{R}_ϕ is quasi-Markovian with respect to μ , then*

$$h_\mu(\phi) \leq \int \log J_\phi d\mu.$$

We also obtain the reverse inequality if the invariant probability measure is absolutely continuous with respect to Lebesgue measure.

Theorem B. *Let ϕ be a C^1 piecewise expanding map with bounded distortion. If \mathcal{R}_ϕ is quasi-Markovian with respect to an absolutely continuous invariant probability measure μ such that $H_\mu(\mathcal{R}_\phi) < \infty$, then*

$$h_\mu(\phi) = \int \log J_\phi d\mu.$$

We shall refer to the conclusion of Theorem B as the *entropy formula* for μ .

The same conclusion of Theorem B has been drawn in [6, Proposition 4.1], under the stronger Markovian assumption of maps with *full branches*. Also in [17, Theorem 1], in the more general setting of a measurable transformation and a conformal reference measure, replacing our quasi-Markovian condition by

$$\sup_{n \geq 1} \left| \int \log m(\phi^n(R^n(x))) d\mu(x) \right| < \infty, \quad (2)$$

where μ is absolutely continuous with respect to the reference measure m ; see (2.4) in [17]. Even though our quasi-Markovian condition introduced above has the same flavor of (2), in practice our condition is easier to deal with. Actually, we were able to check it for the family of tent maps that we introduce in Subsection 1.5 below and we were not able to check condition (2) for that family.

Our next goal is to establish some useful criterium for obtaining the quasi-Markovian property for the partition of a piecewise expanding map with respect to an absolutely continuous invariant probability measure. We define the *singular set* of a piecewise expanding map ϕ as

$$\mathcal{S}_\phi = \overline{\bigcup_{R \in \mathcal{R}_\phi} \partial R},$$

where ∂ stands for the boundary and bar for the closure of a set. Notice that when the partition \mathcal{R}_ϕ is finite the singular set \mathcal{S}_ϕ is a finite union of $(d-1)$ -dimensional submanifolds of \mathbb{R}^d . We say that a piecewise expanding maps ϕ *behaves as a power of the distance* close to \mathcal{S}_ϕ if there exist constants $B, \beta > 0$ such that

$$(S1) \quad \|D\phi(x)\| \leq \frac{B}{\text{dist}(x, \mathcal{S}_\phi)^\beta};$$

$$(S2) \quad \log \frac{\|D\phi(x)^{-1}\|}{\|D\phi(y)^{-1}\|} \leq \frac{B}{\text{dist}(x, \mathcal{S}_\phi)^\beta} \text{dist}(x, y);$$

for every $x, y \in M \setminus \mathcal{S}_\phi$ with $\text{dist}(x, y) < \text{dist}(x, \mathcal{S}_\phi)/2$.

In Proposition 3.4 we establish a criterium for the quasi-Markovian property of the partition associated to a piecewise expanding map behaving as a power of the distance close to the singular set. Using that criterium we easily obtain the next result as a consequence of Theorem B. This will be particularly useful for establishing the entropy formula of the family of tent maps in Subsection 1.5.

Theorem C. *Let ϕ be a C^1 piecewise expanding map with bounded distortion for which (S1)-(S2) hold, and let μ be an ergodic absolutely continuous invariant probability measure for ϕ such that $H_\mu(\mathcal{R}_\phi) < \infty$. If $\log \text{dist}(\cdot, \mathcal{S}_\phi) \in L^p(m)$ and $d\mu/dm \in L^q(m)$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$, then*

$$h_\mu(\phi) = \int \log J_\phi d\mu.$$

Theorems A and B are proved in Section 2, Theorem C is proved in Section 3.

1.3. Continuity of entropy. Now we consider families of piecewise expanding maps. Let I be a metric space and $(\phi_t)_{t \in I}$ be a family of C^1 piecewise expanding maps $\phi_t : \Omega \rightarrow \Omega$, where Ω is a compact subset of \mathbb{R}^d , for some $d \geq 1$. For simplicity, for each $t \in I$ denote

by \mathcal{R}_t the partition of Ω associated to the piecewise expanding map ϕ_t . Our next result gives sufficient conditions for the continuity of the entropy of the absolutely continuous invariant probability measure.

Theorem D. *Let $(\phi_t)_{t \in I}$ be a family of C^1 piecewise expanding maps with bounded distortion such that each ϕ_t has an absolutely continuous invariant probability measure μ_t for which $H_{\mu_t}(\mathcal{R}_t) < \infty$ and the entropy formula holds. Assume that there are $1 < p, q \leq \infty$ with $1/p + 1/q < 1$ such that*

- (1) $d\mu_t/dm$ is uniformly bounded in $L^p(m)$ and depends continuously on $t \in I$ in $L^1(m)$;
- (2) $\log J_{\phi_t} \in L^q(m)$ and $\log J_{\phi_t}$ depends continuously on $t \in I$ in $L^1(m)$.

Then $h_{\mu_t}(\phi_t)$ depends continuously on $t \in I$.

The proof of this theorem will be given in Section 4 and uses the entropy formula of Theorem B. In the next subsections we introduce more particular settings where the previous results above can be applied.

1.4. Maps with large branches. A special class of piecewise expanding maps $\phi : \Omega \rightarrow \Omega$ with bounded distortion for which an absolutely continuous invariant probability measure always exists has been introduced in [1], namely maps that satisfy the following additional condition on the images of the smoothness domains. We say that ϕ has *large branches* if

- (P₃) there are constants $\alpha, \beta > 0$ and for each $R \in \mathcal{R}_\phi$ there is a C^1 unitary vector field X in $\partial\phi(R)^1$ such that:
- (a) the line segments joining each $x \in \partial\phi(R)$ to $x + \alpha X(x)$ are pairwise disjoint contained in $\phi(R)$, and their union form a neighborhood of $\partial\phi(R)$ in $\phi(R)$;
 - (b) for every $x \in \partial\phi(R)$ and $v \in T_x\partial\phi(R) \setminus \{0\}$ the angle $\theta(x, v)$ between v and $X(x)$ satisfies $|\sin \theta(x, v)| \geq \beta$.

In the one-dimensional case, condition (P₃)(a) is obviously satisfied when the elements in \mathcal{R}_ϕ are intervals whose images have sizes uniformly bounded away from zero.

Remark 1.1. Condition (P₃)(b) makes no sense in dimension one, but in practice we can assume the optimal value $\beta = 1$; see [7, Remark 3.2].

From [1, Theorem 5.2] we know that if $\phi : \Omega \rightarrow \Omega$ is a C^2 piecewise expanding map for which conditions (P₁)-(P₃)² hold with

$$\sigma(1 + 1/\beta) < 1, \tag{*}$$

then ϕ has a finite number of absolutely continuous invariant probability measures. Under these assumptions, we also have that the density of any such measure belongs to $BV(\Omega)$, the space of bounded variation functions; see [7, Corollary 3.4]. Assuming $\Omega \subset \mathbb{R}^d$, from Sobolev Inequality we deduce that $BV(\Omega)$ is contained in $L^{d/(d-1)}(m)$; see e.g. [20, Theorem 1.28]. Since $1/d + (d-1)/d = 1$, using Theorem C we easily get the following result.

¹ At the points $x \in \partial\phi(R)$ where $\partial\phi(R)$ is not smooth the vector $X(x)$ is a common C^1 extension of X restricted to each $(d-1)$ -dimensional smooth component of $\partial\phi(R)$ having x in its boundary. The tangent space at any such point is the union of the tangent spaces to the $(d-1)$ -dimensional smooth components that point belongs to.

² Our bounded distortion condition (P₂) is slightly different from the one used in [1]. However, in Lemma 5.1 we show that (P₂) implies the bounded distortion condition in [1], exactly with the same constant $\Delta > 0$.

Corollary E. *Let $\phi : \Omega \rightarrow \Omega$, with $\Omega \subset \mathbb{R}^d$, be a C^1 piecewise expanding map with bounded distortion and large branches for which (S1)-(S2) and (*) hold. If $\log \text{dist}(\cdot, \mathcal{S}_\phi) \in L^d(m)$ and μ is an ergodic absolutely continuous invariant probability measure for ϕ such that $H_\mu(\mathcal{R}_\phi) < \infty$, then*

$$h_\mu(\phi) = \int \log J_\phi d\mu.$$

In addition, [7, Theorem A] asserts that if $(\phi_t)_{t \in I}$ is a family of C^1 piecewise expanding maps $\phi_t : \Omega \rightarrow \Omega$ satisfying properties (1) and (2) of Theorem F below, then $(\phi_t)_{t \in I}$ is *statistically stable*: for any sequence $(t_n)_n$ in I converging to $t_0 \in I$ and any sequence of ergodic absolutely continuous ϕ_{t_n} -invariant probability measures $(\mu_{t_n})_n$, any accumulation point of the densities $d\mu_{t_n}/dm$ must converge in the L^1 -norm to the density of an absolutely continuous ϕ_{t_0} -invariant probability measure. Obviously, when each ϕ_t has a unique (hence ergodic) absolutely continuous invariant probability measure μ_t , then statistical stability means continuity of $d\mu_t/dm$ in the L^1 -norm with $t \in I$. In the next result we give sufficient conditions for the continuity of the entropy of the absolutely continuous invariant probability measure in the setting of piecewise expanding maps with large branches.

Theorem F. *Let $(\phi_t)_{t \in I}$ be a family of C^2 piecewise expanding maps with bounded distortion and large branches such that each ϕ_t has a unique absolutely continuous invariant probability measure μ_t for which $H_{\mu_t}(\mathcal{R}_t) < \infty$ and the entropy formula holds. Assume that*

- (1) *there are $0 < \lambda < 1$ and $K > 0$ such that for each $t \in I$*

$$\sigma_t \left(1 + \frac{1}{\beta_t}\right) \leq \lambda \quad \text{and} \quad \Delta_t + \frac{1}{\alpha_t \beta_t} + \frac{\Delta_t}{\beta_t} \leq K,$$

where $\sigma_t, \Delta_t, \alpha_t, \beta_t$ are constants such that (P₁)-(P₃) hold for ϕ_t ;

- (2) *$f \circ \phi_t$ depends continuously on $t \in I$ in $L^d(m)$, for each continuous $f : \Omega \rightarrow \mathbb{R}$;*
(3) *$\log J_{\phi_t} \in L^q(m)$ for some $q > d$ and $\log J_{\phi_t}$ depends continuously on $t \in I$ in $L^1(m)$.*

Then $h_{\mu_t}(\phi_t)$ depends continuously on $t \in I$.

The C^2 differentiability assumed in this last result is due to the bounded distortion condition considered in [1]. The proof of Theorem F will be given in Section 5.

1.5. Tent maps. Here we consider some special families of piecewise expanding maps, the first one the family of two-dimensional tent maps introduced in [31], and the second one an analogous family of interval maps. The main results of this section will be obtained as an application of the previous results for piecewise expanding maps.

1.5.1. Two-dimensional maps. Consider the triangles

$$\mathcal{T}_0 = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\} \tag{3}$$

and

$$\mathcal{T}_1 = \{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq 2 - x\}.$$

For each $0 < t \leq 1$, define the map $T_t : \mathcal{T} \rightarrow \mathcal{T}$ on the triangle $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ by

$$T_t(x, y) = \begin{cases} (t(x+y), t(x-y)), & \text{if } (x, y) \in \mathcal{T}_0; \\ (t(2-x+y), t(2-x-y)), & \text{if } (x, y) \in \mathcal{T}_1. \end{cases} \tag{4}$$

The domains \mathcal{T}_0 and \mathcal{T}_1 are separated by the straight line segment

$$\mathcal{C} = \{(x, y) \in \mathcal{T} : x = 1\} \quad (5)$$

that we call the *critical set* of T_t .

As shown in [33], the map T_1 displays the same properties of the well-known one-dimensional tent map defined for $x \in [0, 2]$ as $x \mapsto 2 - 2|x - 1|$. Among them, the consecutive pre-images $\{T_1^{-n}(\mathcal{C})\}_{n \in \mathbb{N}}$ of the critical line \mathcal{C} define a sequence of partitions of \mathcal{T} whose diameter tends to zero as n goes to infinity. This enables us to conjugate T_1 to a one sided shift with two symbols, from which it easily follows that T_1 is transitive in \mathcal{T} . Furthermore, the Lyapunov exponent of any point in \mathcal{T} whose orbit does not hit the critical line is positive (actually, it coincides with $1/2 \log 2$) in every nonzero direction. Finally, there is a (unique) absolutely continuous invariant probability measure for T_1 ; see [33] for details.

The results obtained in [33] for $t = 1$ have been extended to a larger set of parameters. More precisely, it was proved in [32] that for each $t \in [\tau, 1]$, with $\tau = \frac{1}{\sqrt{2}}(\sqrt{2} + 1)^{\frac{1}{4}} \approx 0.882$, the map T_t exhibits a *strange attractor* $A_t \subset \mathcal{T}$, i.e. T_t is (strongly) transitive in A_t , the periodic orbits are dense in A_t , and there exists a dense orbit in A_t with two positive Lyapunov exponents. Furthermore, A_t supports a unique absolutely continuous invariant probability measure μ_t . As shown in [7], these measures μ_t depend continuously on $t \in [\tau, 1]$ in a strong sense: their densities vary continuously with the parameter in the norm of $L^1(m)$. Here we show that the entropy with respect to the absolutely continuous invariant probability measure depends continuously on the parameter as well.

Theorem G. *Each T_t has a unique absolutely continuous invariant probability measure μ_t depending continuously on $t \in [\tau, 1]$. Moreover, the entropy formula holds for μ_t and $h_{\mu_t}(T_t)$ depends continuously on $t \in [\tau, 1]$.*

The existence and uniqueness of the absolutely continuous invariant probability measure μ_t has already been proved in [32], and its continuous dependence (in a strong sense) on the parameter t proved in [7].

1.5.2. *One-dimensional maps.* Though easier to deal with, but not following as an immediate consequence of the results for the family of two-dimensional tent maps presented above, we can as well obtain similar conclusions for the one-dimensional family of tent maps $T_t : [0, 2] \rightarrow [0, 2]$, defined for $1 < t \leq 2$ and $x \in [0, 2]$ as

$$T_t(x) = \begin{cases} tx, & \text{if } 0 \leq x \leq 1; \\ t(2 - x), & \text{if } 1 \leq x \leq 2. \end{cases} \quad (6)$$

Theorem H. *Each T_t has a unique absolutely continuous invariant probability measure μ_t depending continuously on $t \in (1, 2]$. Moreover, the entropy formula holds for μ_t and $h_{\mu_t}(T_t)$ depends continuously on $t \in (1, 2]$.*

The existence of an absolutely continuous invariant probability measure for the maps in this family follows as an easy consequence of [25, Theorem 1]. We did not find in the literature any reference to the continuity of this measure or its entropy with the parameter.

2. ENTROPY FORMULA

In this section we prove Theorem A and Theorem B. Let ϕ be a C^1 piecewise expanding map with bounded distortion. For simplicity, denote \mathcal{R}_ϕ by \mathcal{R} and, for each $n \in \mathbb{N}$, let \mathcal{R}^n be as in (1). Condition (P₁) implies that

$$\text{diam}(\mathcal{R}^n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It follows from Kolmogorov-Sinai Theorem that

$$h_\mu(\phi) = h_\mu(\phi, \mathcal{R}) \tag{7}$$

for any ϕ -invariant measure μ such that \mathcal{R} is a μ mod 0 partition of Ω . Moreover, assuming $H_\mu(\mathcal{R}) < \infty$, Shanon-McMillan-Breiman Theorem gives that

$$h_\mu(\mathcal{R}, x) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu(R^n(x))$$

is well defined for μ almost every $x \in \Omega$, and

$$h_\mu(\phi, \mathcal{R}) = \int h_\mu(\mathcal{R}, x) d\mu.$$

Together with (7) this gives

$$h_\mu(\phi) = \int h_\mu(\mathcal{R}, x) d\mu. \tag{8}$$

In the case that μ being ergodic, Shanon-McMillan-Breiman Theorem also gives that

$$h_\mu(\phi) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu(R^n(x)) \tag{9}$$

for μ almost every x . Notice that (8) holds whenever μ is a ϕ -invariant probability measure with $H_\mu(\mathcal{R}) < \infty$ and \mathcal{R} is a μ mod 0 partition of Ω . Moreover, (9) holds when μ is additionally ergodic.

Now we give a simple bounded distortion result that will be used in the proofs of both inequalities that yield the entropy formula.

Lemma 2.1. *There is $C > 0$ such that for all $R_n \in \mathcal{R}^n$ and all $x, y \in R_n$*

$$\frac{1}{C} \leq \frac{J_{\phi^n}(x)}{J_{\phi^n}(y)} \leq C.$$

Proof. For all $x, y \in R_n$ we may write

$$\begin{aligned} \log \frac{J_{\phi^n}(x)}{J_{\phi^n}(y)} &= \sum_{j=0}^{n-1} \log \frac{J_\phi(\phi^j(x))}{J_\phi(\phi^j(y))} \\ &\leq \sum_{j=0}^{n-1} \Delta \|\phi^j(x) - \phi^j(y)\| \\ &\leq \sum_{j=0}^{n-1} \Delta \sigma^{n-j} \|\phi^n(x) - \phi^n(y)\|, \quad \text{by (P}_1\text{)} \\ &\leq \sum_{j=0}^{n-1} \Delta \sigma^{n-j} \text{diam}(\Omega). \end{aligned}$$

Taking exponentials and using the symmetry on the roles of x and y we easily finish the proof. \square

Now we split the proofs of Theorems A and B into the next two inequalities.

2.1. First inequality. In this subsection we complete the proof of Theorem A, following some ideas from the proof of [22, Theorem 1]. Take μ a ϕ -invariant probability measure such that $H_\mu(\mathcal{R}_\phi) < \infty$ and assume that \mathcal{R}_ϕ is quasi-Markovian with respect to μ .

Consider first the case in which μ is an ergodic probability measure. Assuming by contradiction that the conclusion of Theorem A does not hold, choose real numbers α, β such that

$$h_\mu(\phi) > \alpha > \beta > \int \log J_\phi d\mu. \quad (10)$$

It follows from (9) and (10) that there exists a set $E_1 \subset \Omega$ with $\mu(E_1) = 1$ such that for every $x \in E_1$ there exists $k_1 = k_1(x) \in \mathbb{N}$ for which

$$\mu(R^n(x)) \leq e^{-\alpha n}, \quad \forall n \geq k_1. \quad (11)$$

Moreover, by Birkhoff's Ergodic Theorem we have

$$\frac{1}{n} \log J_{\phi^n}(x) = \frac{1}{n} \sum_{j=0}^{n-1} \log J_\phi(\phi^j(x)) \rightarrow \int \log J_\phi d\mu, \quad \text{as } n \rightarrow \infty,$$

which then implies that there exists a set $E_2 \subset \Omega$ with $\mu(E_2) = 1$ such that for every $x \in E_2$ there exists $k_2 = k_2(x) \in \mathbb{N}$ for which

$$J_{\phi^n}(x) \leq e^{\beta n}, \quad \forall n \geq k_2. \quad (12)$$

On the other hand, there exist $\eta > 0$ and a set $E_3 \subset \Omega$ with $\mu(E_3) = 1$ such that every $x \in E_3$ there are infinitely many values of $n \in \mathbb{N}$ for which

$$m(\phi^n(R^n(x))) \geq \eta. \quad (13)$$

Using Lemma 2.1, for each $n \in \mathbb{N}$ we have

$$\begin{aligned} m(\phi^n(R^n(x))) &= \int_{R^n(x)} J_{\phi^n}(y) dm(y) \\ &= \int_{R^n(x)} \frac{J_{\phi^n}(y)}{J_{\phi^n}(x)} J_{\phi^n}(x) dm(y) \\ &\leq C J_{\phi^n}(x) m(R^n(x)). \end{aligned} \quad (14)$$

Now take an arbitrary $\ell \in \mathbb{N}$. Given any $x \in E_1 \cap E_2 \cap E_3$ choose $n(x) \geq \max\{k_1, k_2, \ell\}$ such that (13) holds. It follows from (12), (13) and (14) that

$$\frac{C}{\eta} m(R^{n(x)}(x)) \geq e^{-\beta n(x)},$$

which together with (11) gives

$$\mu(R^{n(x)}(x)) \leq e^{-n(x)(\alpha-\beta)} \frac{C}{\eta} m(R^{n(x)}(x)) \leq e^{-\ell(\alpha-\beta)} \frac{C}{\eta} m(R^{n(x)}(x)). \quad (15)$$

Defining

$$\mathcal{Q} = \{R_{n(x)}(x) : x \in E_1 \cap E_2 \cap E_3\}$$

we have that \mathcal{Q} is a μ mod 0 partition of Ω . As two elements in \mathcal{Q} are either disjoint or one contains the other, we can extract a μ mod 0 subcover $\tilde{\mathcal{Q}}$ of Ω by pairwise disjoint sets. From (15) we get

$$1 = \mu(\Omega) \leq \sum_{Q \in \tilde{\mathcal{Q}}} \mu(Q) \leq e^{-\ell(\alpha-\beta)} \frac{C}{\eta} \sum_{Q \in \tilde{\mathcal{Q}}} m(Q) \leq e^{-\ell(\alpha-\beta)} \frac{C}{\eta}.$$

Since $\alpha > \beta$ and ℓ can be taken arbitrarily large, this gives a contradiction.

Assume now that μ is not an ergodic probability measure. By the Ergodic Decomposition Theorem, there exists a probability measure θ on the Borel sets of $E_\phi(\Omega)$ such that for any measurable function $f : \Omega \rightarrow \mathbb{R}$ we have

$$\int_{\Omega} f d\mu = \int_{E_\phi(\Omega)} \int_{\Omega} f d\nu d\theta(\nu),$$

where $E_\phi(\Omega)$ stands for the set of ϕ -invariant ergodic probability measures on the Borel sets of Ω , endowed with the weak* topology. As we are assuming \mathcal{R} quasi-Markovian with respect to μ , it follows from the Ergodic Decomposition Theorem that \mathcal{R} must be quasi-Markovian with respect to any measure ν in the ergodic decomposition of μ . Moreover, by the concavity of the function $-x \log x$ and Jensen Inequality, we have

$$H_\mu(\mathcal{R}) \geq \int_{E_\phi(\Omega)} H_\nu(\mathcal{R}) d\theta(\nu).$$

As we are assuming $H_\mu(\mathcal{R}) < \infty$, we also have $H_\nu(\mathcal{R}) < \infty$ for θ almost every measure ν in the ergodic decomposition of μ . Hence, by the case already seen for an ergodic measure, we can write

$$\begin{aligned} h_\mu(\phi) &= \int_{\Omega} h_\mu(\mathcal{R}, x) d\mu \\ &= \int_{E_\phi(\Omega)} \int_{\Omega} h_\mu(\mathcal{R}, x) d\nu d\theta(\nu) \\ &\leq \int_{E_\phi(\Omega)} \int_{\Omega} \log J_\phi d\nu d\theta(\nu) \\ &= \int_{\Omega} \log J_\phi d\mu. \end{aligned}$$

2.2. Second inequality. Let us now complete the proof of Theorem B. Assume now that μ is an absolutely continuous ϕ -invariant probability measure such that $H_\mu(\mathcal{R}_\phi) < \infty$ and \mathcal{R}_ϕ is quasi-Markovian with respect to μ . By Theorem A it is enough to show that

$$h_\mu(\phi) \geq \int_{\Omega} \log(J_\phi) d\mu. \quad (16)$$

Consider ρ the density of μ with respect to m , given by Radon-Nikodym Theorem. We have for μ almost every $x \in \Omega$

$$\lim_{n \rightarrow \infty} \frac{\mu(R^n(x))}{m(R^n(x))} = \rho(x) > 0. \quad (17)$$

Since

$$\frac{1}{n} \log \mu(R^n(x)) = \frac{1}{n} \log m(R^n(x)) + \frac{1}{n} \log \frac{\mu(R^n(x))}{m(R^n(x))},$$

it follows from (17) that

$$h_\mu(\mathcal{R}, x) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log(m(R^n(x))), \quad (18)$$

for μ almost every $x \in \Omega$. We may write

$$m(\Omega) \geq m(\phi^n(R^n(x))) = \int_{R^n(x)} J_{\phi^n}(y) dm(y) = \int_{R^n(x)} \frac{J_{\phi^n}(y)}{J_{\phi^n}(x)} J_{\phi^n}(x) dm(y). \quad (19)$$

It follows from Lemma 2.1 and (19) that

$$m(\Omega) \geq \frac{1}{C} J_{\phi^n}(x) m(R^n(x))$$

which then implies

$$-\frac{1}{n} \log m(R^n(x)) \geq -\frac{\log C + \log m(\Omega)}{n} + \frac{1}{n} \log J_{\phi^n}(x). \quad (20)$$

Using (8), (18) and (20) we easily deduce that

$$h_\mu(\phi) \geq \int \lim_{n \rightarrow \infty} \frac{1}{n} \log(J_{\phi^n}(x)) d\mu = \int \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(J_{\phi^i}(x)) d\mu.$$

Finally, using Birkhoff's Ergodic Theorem we obtain (16).

3. QUASI-MARKOVIAN PROPERTY

The goal of this section is to prove Theorem C. This will be obtained as a consequence of Theorem B and Proposition 3.4 below, which gives a useful criterium for obtaining the quasi-Markovian property. We start by recalling some general facts from [3] about maps (not necessarily piecewise expanding) which are non-uniformly expanding and have slow recurrence to a singular set.

Let M be a compact manifold, \mathcal{S} a compact subset of M and $f : M \setminus \mathcal{S} \rightarrow M$ a C^1 map. Typically, \mathcal{S} is a set of points where the map f fails to be differentiable or even continuous, or even if f is differentiable its derivative is not an isomorphism. For the sake of completeness, let us refer that in the general setting of [3], where maps with critical sets are allowed, condition (S1) introduced in Subsection 1.2 above to express that f behaves as a power of the distance close to \mathcal{S} needs to be replaced by the following stronger condition: there are $B, \beta > 0$ such that

$$(S1^*) \quad \frac{1}{B} \text{dist}(x, \mathcal{S})^\beta \leq \frac{\|Df(x)v\|}{\|v\|} \leq B \text{dist}(x, \mathcal{S})^{-\beta},$$

for every $v \in T_x M$ and $x, y \in M \setminus \mathcal{S}$ with $\text{dist}(x, y) < \text{dist}(x, \mathcal{S})/2$. However, notice that for piecewise expanding maps the left hand side of (S1*) is trivially satisfied, and so (S1*) coincides with (S1) in this setting. Related to this, see Remark 3.2 below.

Given $\delta > 0$ and $x \in M \setminus \mathcal{S}$ we define the δ -truncated distance from x to \mathcal{S} as

$$\text{dist}_\delta(x, \mathcal{S}) = \begin{cases} \text{dist}(x, \mathcal{S}), & \text{if } \text{dist}(x, \mathcal{S}) < \delta; \\ 1, & \text{otherwise.} \end{cases}$$

We say that f is *non-uniformly expanding (NUE)* on a set $H \subset M$ if there is some $c > 0$ such that for all $x \in H$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < -c. \quad (21)$$

Moreover, we say that f has *slow recurrence (SR)* to \mathcal{S} on H if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in H$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(f^j(x), \mathcal{S}) \leq \varepsilon.$$

The next result is a consequence of Lemmas 5.2 and 5.4 in [3], and will be used to prove a useful criterium for a partition of a piecewise expanding map to be quasi-Markovian.

Lemma 3.1. *Let $f : M \setminus \mathcal{S} \rightarrow M$ be a C^2 map such that f behaves as a power of the distance close to \mathcal{S} . If NUE and SR hold for a set $H \subset M$, then there exists $\delta_1 > 0$ such that for all $x \in H$ there are infinitely many $n \in \mathbb{N}$ and an open neighborhood $V_n(x)$ of x which is mapped by f^n diffeomorphically onto the ball of radius δ_1 around $f^n(x)$.*

Though the results in [3] are stated for boundaryless manifolds, it is not difficult to see that the conclusion of Lemma 3.1 still holds for a manifold M with boundary, provided the boundary of M is included in the singular set \mathcal{S} .

Remark 3.2. In [3], a third condition (S3) is considered in the definition of a map behaving as a power of the distance close to the singular set. However, that condition (S3) is only used in the proof of [3, Corollary 5.3] to deduce a bounded distortion property on the sets $V_n(x)$. Here we do not need that property.

3.1. Piecewise expanding maps. Let us now go back to piecewise expanding maps. As observed above, in this setting we have condition (S1*) equivalent to (S1).

Lemma 3.3. *Let $\phi : \Omega \rightarrow \Omega$ be a C^1 piecewise expanding map for which (S1)-(S2) hold. If $\log \text{dist}(\cdot, \mathcal{S}_\phi) \in L^p(m)$ and μ is an ergodic absolutely continuous invariant probability measure for ϕ such that $d\mu/dm \in L^q(m)$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$, then ϕ has slow recurrence to \mathcal{S}_ϕ on a subset of Ω with full μ measure.*

Proof. Define for $x \in \Omega \setminus \mathcal{S}_\phi$

$$\xi(x) = -\log \text{dist}(x, \mathcal{S}_\phi).$$

Since we are assuming that $\log \text{dist}(\cdot, \mathcal{S}_\phi) \in L^p(m)$ and $d\mu/dm \in L^q(m)$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$, then Hölder Inequality gives that $\xi \in L^1(\mu)$. Hence,

$$\lim_{\delta \rightarrow 0^+} \int_{\{\xi > -\log \delta\}} \xi d\mu_t = 0.$$

Observing that we have

$$\chi_{\{\xi > -\log \delta\}} \xi = -\log \text{dist}_\delta(\cdot, \mathcal{S}_\phi),$$

where $\chi_{\{\xi > -\log \delta\}}$ denotes the characteristic function of the set $\{\xi > -\log \delta\}$, it easily follows that for all $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\int -\log \text{dist}_\delta(\cdot, \mathcal{S}_\phi) d\mu < \varepsilon.$$

Hence, using the ergodicity of μ , Birkhoff's Ergodic Theorem yields

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(\phi_t^j(x), \mathcal{S}_\phi) = \int -\log \text{dist}_\delta(\cdot, \mathcal{S}_\phi) d\mu_t < \varepsilon$$

for μ almost every $x \in \Omega$. \square

Proposition 3.4. *Let ϕ be a C^1 piecewise expanding map for which (S1)-(S2) hold. If $\log \text{dist}(\cdot, \mathcal{S}_\phi) \in L^p(m)$ and μ is an absolutely continuous invariant probability measure for which $d\mu/dm \in L^q(m)$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$, then \mathcal{R}_ϕ is quasi-Markovian with respect to μ .*

Proof. Considering, as before, $R^n(x)$ the element in \mathcal{R}^n containing x , we need to show that there is some constant $\eta > 0$ such that for μ almost every $x \in \Omega$, there are infinitely many values of $n \in \mathbb{N}$ for which

$$m(\phi^n(R^n(x))) \geq \eta.$$

By Lemma 3.3, there is a set $H \subset T$ with full μ measure such that ϕ has slow recurrence to \mathcal{S}_ϕ on H . On the other hand, since ϕ is a piecewise expanding map, then ϕ is clearly non-uniformly expanding on H . Hence, by Lemma 3.1 there exists $\delta_1 > 0$ such that for all $x \in H$ there are infinitely many $n \in \mathbb{N}$ and an open neighborhood $V_n(x)$ of x which is mapped by f^n diffeomorphically onto the ball of radius δ_1 around $f^n(x)$. In such case, each set $V_n(x)$ is necessarily contained in $R^n(x)$, for $V_n(x)$ is mapped by ϕ^n diffeomorphically onto its image. Recalling that this image is the ball of radius $\delta_1 > 0$ around $\phi^n(x)$, we have

$$m(\phi^n(R^n(x))) \geq m(\phi^n(V_n(x))) \geq \pi\delta_1^2,$$

thus having proved that \mathcal{R}_ϕ is quasi-Markovian with respect to μ . \square

4. CONTINUITY OF ENTROPY

In this section we prove Theorem D. This will be obtained as a consequence of Theorem B, together with Lemma 4.1 below. The proof of this lemma would be a straightforward application of Hölder Inequality if the convergence were in the norm of $L^p(m)$ and $1/p + 1/q = 1$. Nevertheless, slightly improving the regularity of functions, we are able to obtain a useful criterium for the case that convergence holds only in the norm of $L^1(m)$.

Lemma 4.1. *Consider $1 < p, q \leq \infty$ with $1/p + 1/q < 1$. Assume that*

- (1) $(f_n)_n$ is a bounded sequence in $L^p(m)$ converging to $f \in L^p(m)$ in $L^1(m)$;
- (2) $(g_n)_n$ is a bounded sequence in $L^q(m)$.

Then

$$\lim_{n \rightarrow \infty} \int (f_n - f)g_n dm = 0.$$

Proof. Take an arbitrary $\epsilon > 0$. Consider $M > 0$ such that $\|f\|_p \leq M$, $\|f_n\|_p \leq M$ and $\|g_n\|_q \leq M$ for all $n \geq 1$, and let $1 \leq r < \infty$ be such that $1/q + 1/r = 1$. Define for each $n \geq 1$

$$B_n = \left\{ x \in \Omega : |f_n(x) - f(x)| > \frac{\epsilon}{2Mm(\Omega)^{1/r}} \right\}.$$

Since $\|f_n - f\|_1 \rightarrow 0$ as $n \rightarrow \infty$, we necessarily have

$$\lim_{n \rightarrow \infty} m(B_n) = 0. \tag{22}$$

We can write

$$\int_{\Omega} |f_n - f| |g_n| dm = \int_{\Omega \setminus B_n} |f_n - f| |g_n| dm + \int_{B_n} |f_n - f| |g_n| dm. \quad (23)$$

For the first term in the sum above we have

$$\int_{\Omega \setminus B_n} |f_n - f| |g_n| dm \leq \frac{\epsilon}{2Mm(\Omega)^{1/r}} \int_{\Omega} |g_n| dm \leq \frac{\epsilon}{2Mm(\Omega)^{1/r}} \|g_n\|_q m(\Omega)^{1/r} < \frac{\epsilon}{2}.$$

Let us see that, for n sufficiently large, the second term in (23) can also be made smaller than $\epsilon/2$. Consider now $1 \leq s < \infty$ such that $1/p + 1/s = 1$. Notice that one necessarily has $s < q$, and so $g_n \in L^q(m) \subset L^s(m)$. Using Hölder inequality we get

$$\int_{B_n} |f_n - f| |g_n| dm \leq \|f_n - f\|_p \|\chi_{B_n} g_n\|_s \leq 2M \|\chi_{B_n} g_n\|_s. \quad (24)$$

Moreover, taking $1 \leq u < \infty$ such that $s/q + 1/u = 1$

$$\|\chi_{B_n} g_n\|_s^s = \int_{B_n} |g_n|^s dm \leq \|\chi_{B_n}\|_u \|g_n^s\|_{q/s} \leq m(B_n)^{1/u} \|g_n\|_q^s,$$

which then gives

$$\|\chi_{B_n} g_n\|_s \leq m(B_n)^{1/(su)} \|g_n\|_q \leq Mm(B_n)^{1/(su)} \quad (25)$$

Hence, using (22), (23), (24) and (25) we easily see that the second term on the right hand side of (23) can be made smaller than $\epsilon/2$ as well, for n sufficiently large. \square

Let us now complete the proof of Theorem D. By Theorem B, it is enough to show that the function

$$I \ni t \longmapsto \int_{\Omega} \log(J_{\phi_t}) d\mu_t$$

is continuous. Let $(t_n)_n$ be an arbitrary sequence in I converging to $t_0 \in I$. Considering

$$\rho_n = \frac{d\mu_{t_n}}{dm},$$

we may write

$$\left| \int_{\Omega} \log(J_{\phi_{t_0}}) d\mu_{t_0} - \int_{\Omega} \log(J_{\phi_{t_n}}) d\mu_{t_n} \right| \leq \left| \int_{\Omega} (\log(J_{\phi_{t_0}}) - \log(J_{\phi_{t_n}})) \rho_n dm \right| + \left| \int_{\Omega} (\rho_0 - \rho_n) \log(J_{\phi_{t_0}}) dm \right|. \quad (26)$$

Using Lemma 4.1 with $f_n = \log(J_{\phi_{t_n}})$ and $g_n = \rho_n$ in the first term of the sum above, and $f_n = \rho_n$ and $g_n = \log(J_{\phi_{t_0}})$ in the second one, it immediately follows that, under the assumptions of Theorem D, both terms in the sum converge to zero when n goes to infinity, thus giving the desired conclusion.

5. MAPS WITH LARGE BRANCHES

In this section we prove Theorem F. Assume that $(\phi_t)_{t \in I}$ is a family of C^2 piecewise expanding maps for which the assumptions of Theorem F hold. In particular, there is $q > d$ for which $\log J_{\phi_t} \in L^q(m)$ and $\log J_{\phi_t}$ depends continuously on $t \in I$ in $L^1(m)$. We are going to prove that, in this setting, the assumptions of Theorem D are verified. In practice, we only have to show that if we take $\rho_t = d\mu_t/dm$, then there is some $1 < p \leq \infty$ with $1/p + 1/q < 1$ such that ρ_t is uniformly bounded in $L^p(m)$ and ρ_t depends continuously on $t \in I$ in $L^1(m)$. We start with a preliminary result whose conclusion gives the bounded distortion condition used in [7, Theorem A].

Lemma 5.1. *Let ϕ be a C^2 piecewise expanding map. If the bounded distortion condition (P_2) holds, then for all $R \in \mathcal{R}_\phi$ and all $x \in \text{int } \phi(R)$ we have*

$$\frac{\|D(J_\phi \circ \phi_R^{-1})(x)\|}{|J_\phi \circ \phi_R^{-1}(x)|} \leq \Delta,$$

where $\Delta > 0$ is the constant in (P_2) .

Proof. It is enough to show that for all $x \in \text{int } \phi(R)$ and all $1 \leq j \leq d$ we have

$$\left| \frac{\frac{\partial}{\partial x_j} (J_\phi \circ \phi_R^{-1})(x)}{J_\phi \circ \phi_R^{-1}(x)} \right| \leq \Delta.$$

In fact,

$$\begin{aligned} & \left| \frac{\frac{\partial}{\partial x_j} (J_\phi \circ \phi_R^{-1})(x)}{J_\phi \circ \phi_R^{-1}(x)} \right| \\ &= \left| \frac{\partial}{\partial x_j} \log(J_\phi \circ \phi_R^{-1})(x) \right| \\ &= \left| \lim_{h \rightarrow 0} \frac{1}{h} (\log J_\phi(\phi_R^{-1}(x_1, \dots, x_j + h, \dots, x_d)) - \log J_\phi(\phi_R^{-1}(x_1, \dots, x_j, \dots, x_d))) \right| \\ &= \left| \lim_{h \rightarrow 0} \frac{1}{h} \left(\log \frac{J_\phi(\phi_R^{-1}(x_1, \dots, x_j + h, \dots, x_d))}{J_\phi(\phi_R^{-1}(x_1, \dots, x_j, \dots, x_d))} \right) \right| \end{aligned} \quad (27)$$

Now, by (P_2) we have

$$\log \frac{J_\phi(\phi_R^{-1}(x_1, \dots, x_j + h, \dots, x_d))}{J_\phi(\phi_R^{-1}(x_1, \dots, x_j, \dots, x_d))} \leq \Delta|h|$$

This obviously implies that the expression in (27) is bounded by Δ . \square

Now, using Lemma 5.1, we easily deduce that, under conditions (1) and (2) of Theorem F, the assumptions of [7, Theorem A] are verified. Hence, as we are assuming uniqueness of the absolutely continuous invariant probability measure, using [7, Theorem A] we obtain in this particular case that ρ_t depends continuously on $t \in I$ in $L^1(m)$.

Let us finally prove that there is some $1 < p \leq \infty$ with $1/p + 1/q < 1$ such that ρ_t is uniformly bounded in $L^p(m)$. Under assumption (1) of Theorem F, it follows from [7,

Corollary 3.4] that the density ρ_t of the absolutely continuous invariant probability measure μ_t is a function with (uniformly) bounded variation. Actually, we have

$$\text{var}(\rho_t) \leq K_1, \quad \text{with} \quad K_1 = \sum_{n=0}^{\infty} \lambda^n K,$$

where K is the constant in condition (2) of Theorem F. By Sobolev's Inequality (see e.g. [20, Theorem 1.28]) there is a constant $C > 0$ (only depending on the dimension d) such that

$$\|\rho_t\|_p \leq C \text{var}(\rho_t), \quad \text{with} \quad p = \frac{d}{d-1}. \quad (28)$$

This gives that ρ_t is uniformly bounded in $L^p(m)$. Finally, observing that

$$\frac{1}{p} + \frac{1}{q} = \frac{d-1}{d} + \frac{1}{q} < \frac{d-1}{d} + \frac{1}{d} = 1,$$

we conclude the proof of Theorem F.

6. TENT MAPS

In this section we prove Theorems G and H. Both proofs are based in the simple fact that if, for t belonging to some set of parameters I , the map T_t has a unique absolutely continuous invariant probability measure μ_t which also happens to be the unique absolutely continuous invariant probability measure for a certain power T_t^n , and the family $\{T_t^n\}_{t \in I}$ is in the conditions of Corollary E and Theorem F, then the entropy formula still holds for T_t , and $h_{\mu_t}(T_t)$ depends continuously on $t \in I$. This is actually a straightforward consequence of the following well-known formulas establishing that for each $t \in I$ one has

$$h_{\mu_t}(T_t) = \frac{1}{n} h_{\mu_t}(T_t^n)$$

and

$$\int \log J_{T_t^n} d\mu_t = n \int \log J_{T_t} d\mu_t,$$

6.1. Two-dimensional tents. In this section we obtain Theorem G as a consequence of Theorem F. As assumption (1) in Theorem F does not hold for T_t , we need to take some iterate greater than one. This has already been considered in [7] for obtaining the statistical stability, and T_t^6 is actually enough. As proved in [32, Theorem 1.1], each T_t is *strongly transitive*, meaning that every non-empty open set becomes the whole attractor under a finite number of iterations by T_t . This in particular implies that the absolutely continuous invariant probability measure μ_t for T_t must be unique and ergodic and the strongly transitive attractor of T_t mentioned above coincides with the support of μ_t . Moreover, for any $t \in [\tau, 1]$, any power of T_t is also strongly transitive in the support of μ_t from which we deduce that any power of T_t has a unique ergodic absolutely continuous invariant probability measure as well, which must necessarily coincide with μ_t . All these facts can be checked in [32].

We are going to see that the family $(T_t^6)_{t \in [\tau, 1]}$ is in the conditions of Corollary E and Theorem F. Observe that as for each $t \in [\tau, 1]$ the partition into smoothness domains of T_t^6 is finite, then it necessarily has finite μ_t entropy. Moreover, from [7, Section 4] we know that each $T_t^6 : \mathcal{T} \rightarrow \mathcal{T}$ is a piecewise expanding map with bounded distortion and large

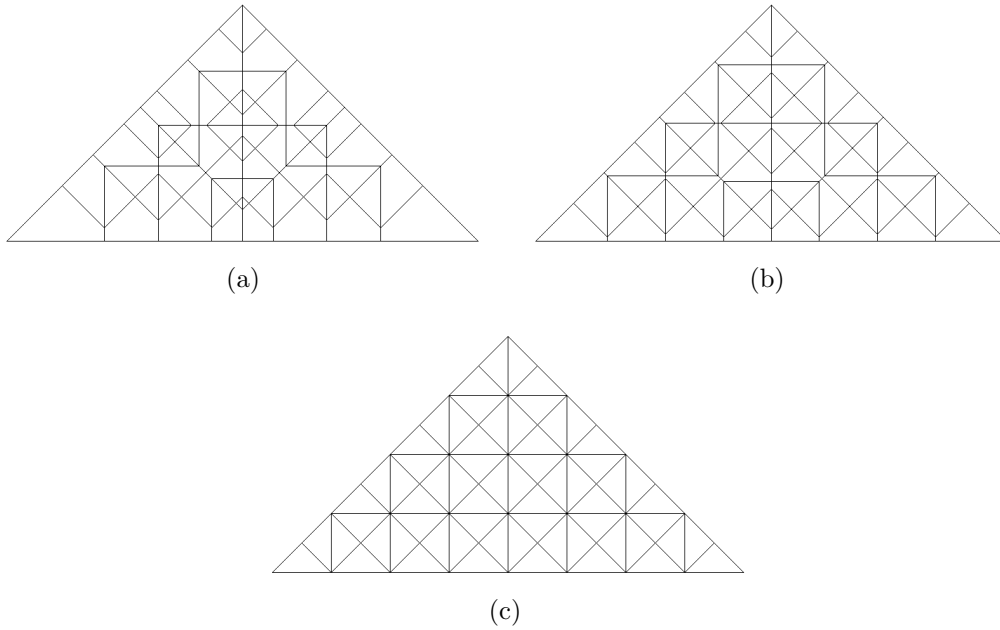


FIGURE 1. Domains of smoothness for T_t^6 : (a) $t = \tau$ (b) $t = 0.95$ (c) $t = 1$

branches satisfying conditions (1) and (2) in the statement of Theorem F. Additionally, since each map T_t^6 is piecewise linear with the slopes and smoothness domains continuously depending on the parameter t (see Figure 1), it is not difficult to see that $\log(J_{T_t^6}) \in L^\infty(m)$ and, moreover, $\log(J_{T_t^6})$ depends continuously on $t \in [\tau, 1]$ in the norm of $L^1(m)$. This is the content of condition (3) with $q = \infty$ in the statement of Theorem F.

It remains to check that the assumptions of Corollary E hold for our family of tent maps. First of all, observe that condition (*) is part of condition (1) in Theorem F, and so it is satisfied. Considering \mathcal{S}_t the singular set of T_t , in this case we have

$$\mathcal{S}_t = \mathcal{C}_6 \cup \partial\mathcal{T},$$

where \mathcal{C}_6 is the critical set for T_t^6 and $\partial\mathcal{T}$ is the boundary of \mathcal{T} . Notice that \mathcal{C}_6 is made by a finite number of straight line segments dividing \mathcal{T} into the smoothness domains of T_t^6 ; see Figure 1. Since T_t^6 has constant derivative on each connected component of $\mathcal{T} \setminus \mathcal{S}_t$, then conditions (S1)-(S2) are obviously satisfied. Finally, as \mathcal{S}_t is a finite union of one-dimensional submanifolds of \mathcal{T} , it follows from [2, Proposition 4.1] that $\log \text{dist}(\cdot, \mathcal{S}_t) \in L^2(m)$. Hence, all the assumptions of Corollary E hold for the two-dimensional tent maps.

6.2. One-dimensional tents. Here we prove Theorem H. Let us recall the family of one-dimensional tent maps $\{T_t\}_{t \in (1, 2]}$, with $T_t : [0, 2] \rightarrow [0, 2]$ given by

$$T_t(x) = \begin{cases} tx, & \text{if } 0 \leq x \leq 1; \\ t(2-x), & \text{if } 1 \leq x \leq 2. \end{cases}$$

Our strategy to prove Theorem H is, once again, to use Corollary E and Theorem F.

Let us start by assuming that $t \in (2^{\frac{1}{4}}, 2]$. In this case, it is easy to see that, if $t \in (2^{\frac{1}{2}}, 2]$, then the interval $A_t = [t(2-t), t]$ is the unique attractor for T_t and, moreover, that T_t is

strongly transitive on A_t . On the other hand, if $t \in (2^{\frac{1}{4}}, 2^{\frac{1}{2}}]$, then T_t has a unique attractor, still denoted by A_t , formed by two disjoint subintervals (with a common endpoint, in the case $t = 2^{\frac{1}{2}}$); see [16] or [15] for details. In fact, we have $A_t = A_t^1 \cup A_t^2$ with $T_t(A_t^1) = A_t^2$, and $T_t(A_t^2) = A_t^1$. Moreover, the map T_t^2 is strongly transitive on A_t^1 and on A_t^2 . In any case, the map T_t^5 is transitive on A_t for every $t \in (2^{\frac{1}{4}}, 2]$ (observe that, for instance, this last claim is not true for T_t^6). For each $t \in (1, 2]$, let μ_t be the unique ergodic absolutely continuous invariant measure for T_t ; see [25]. Following the same arguments given for the two-dimensional case we may assert that, for every $t \in (2^{\frac{1}{4}}, 2]$, we have μ_t as the unique ergodic absolutely continuous invariante measure for T_t^5 .

Let us now explain why we choose the fifth power of the maps for parameters in the interval $(2^{\frac{1}{4}}, 2]$. Observe that for each $t \in (1, 2]$ we have $|T_t'| = t$, which then implies that the piecewise expanding condition (P₁) in Subsection 1.1 is satisfied by T_t with $\sigma_t = 1/t$. However, for having condition (*) satisfied we need to take powers of T_t . Recall that in the one-dimensional case we can always assume $\beta_t = 1$; see Remark 1.1. Moreover, for any $k \in \mathbb{N}$, the map T_t^k satisfies condition (P₁) with $\sigma_t^k = 1/t^k$. Now, a straightforward calculation gives that (*) holds for T_t^5 , uniformly in $t \in (2^{\frac{1}{4}}, 2]$.

Now, observe that the singular set \mathcal{S}_t of T_t^5 is formed by a finite number of critical points where the map is not differentiable, together with the boundary points 0 and 2. As T_t^5 has constant derivative on each connected component of $[0, 2] \setminus \mathcal{S}_t$, conditions (S1)-(S2) are trivially satisfied. Finally, it is easy to see that in this one-dimensional setting we have $\log \text{dist}(\cdot, \mathcal{S}_t) \in L^1(m)$. Hence, all the assumptions of Corollary E are satisfied for the maps T_t^5 , with $t \in (2^{\frac{1}{4}}, 2]$. In this way we deduce that the entropy formula holds for μ_t , whenever $t \in (2^{\frac{1}{4}}, 2]$.

Next, observe that since the maps T_t^5 are continuous, then the second condition in Theorem F is trivially satisfied. This fact together with condition (*) allow us to assert that μ_t depends continuously on $t \in (2^{\frac{1}{4}}, 2]$ according to [7, Theorem A]. Furthermore, it is obvious that $\log(J_{T_t^5}) \in L^\infty(m)$ and, moreover, $\log(J_{T_t^5})$ depends continuously on $t \in (2^{\frac{1}{4}}, 2]$ in the norm of $L^1(m)$. This is the content of condition (3) in the statement of Theorem F with $q = \infty$. Hence, Theorem H is proved for parameters $t \in (2^{\frac{1}{4}}, 2]$.

Now we explain how we can extend these ideas to the whole interval of parameters $(1, 2]$. For this we write

$$(1, 2] = \bigcup_{j=0}^{\infty} \mathcal{I}_j, \quad \text{where } \mathcal{I}_j = (2^{\frac{1}{2^{j+2}}}, 2^{\frac{1}{2^j}}],$$

and prove Theorem H for every $t \in \mathcal{I}_j$ and every $j \in \mathbb{N}$. Observe that, for every $j \geq 0$ we have $\mathcal{I}_j \cap \mathcal{I}_{j+1} = (2^{\frac{1}{2^{j+2}}}, 2^{\frac{1}{2^{j+1}}}]$ and therefore the continuity of the entropy on the sequence of parameters $\{\frac{1}{2^j}\}_{j \in \mathbb{N}}$ will also be guaranteed.

Let us briefly describe the dynamics of T_t for parameters $t \in \mathcal{I}_j$. If $t \in \mathcal{I}_j$ is such that $t \leq 2^{\frac{1}{2^{j+1}}}$ then T_t has an attractor A_t formed by 2^{j+1} disjoint pieces and $T_t^{2^{j+1}}$ is strongly transitive on any of these pieces; see [16] or [15] for details. If $t \in \mathcal{I}_j$ is such that $t > 2^{\frac{1}{2^{j+1}}}$ then T_t has an attractor A_t formed by 2^j disjoint pieces and $T_t^{2^j}$ is strongly transitive on any of these pieces. In any case, $T_t^{2^{j+2}+1}$ is transitive on A_t for every $t \in \mathcal{I}_j$ and, moreover, it is easy to see that condition (*) holds for $T_t^{2^{j+2}+1}$, uniformly in \mathcal{I}_j . The rest of the

arguments needed for proving Theorem H for every $t \in \mathcal{I}_j$ follows in the same way as the ones used before for $t \in \mathcal{I}_0 = (2^{\frac{1}{4}}, 2]$.

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