The Topology of Higgs Bundle Moduli Spaces

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To Sofia

Contents

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

The material of Chapter 3 is a revised version of material which has been submitted for the degree of *cand. scient.* at the University of Aarhus, Denmark, 1993, and which has been published as: P. B. Gothen, *The Betti numbers of the moduli space of rank 3 Higgs bundles*, Internat. J. Math. **5** (1994), 861–875.

Summary

We use the interpretation of the moduli space of flat connections on a Riemann surface in terms of Higgs bundles to study the topology of these spaces.

We calculate the Betti numbers of the moduli space of stable Higgs bundles of rank 3 and degree 1 with fixed determinant, which is homeomorphic to the space of representations of a universal central extension of $\pi_1\Sigma$, in $\mathrm{SL}(3,\mathbb{C})$. The calculation is done using Morse-Bott theory. The critical submanifolds are closely related to the moduli spaces of vortex pairs.

We study the moduli space of flat reductive $\mathrm{Sp}(2n,\mathbb{R})$-bundles and, using Higgs bundles, we obtain an easy proof of a Milnor-Wood type inequality. Furthermore, we study the number of connected components of the moduli space in the case $n = 2$.

Chapter 1

Introduction

Let Σ be a closed surface of genus $g \geqslant 2$, and let G be a Lie group. There has been continuing interest in studying the space of representations

$$\mathbf{M} = \mathrm{Hom}(\pi_1\Sigma, G)/G,$$

where G acts by conjugation. From the point of view of gauge theory, $\mathbf{M}$ is the moduli space of flat G-connections modulo gauge equivalence: the flat principal G-bundle associated to a representation $\rho\colon \pi_1\Sigma \to G$ is

$$P = \tilde{\Sigma} \underset{\pi_1\Sigma}{\times} G,$$

where $\tilde{\Sigma}$ is the universal cover of Σ, and $\pi_1\Sigma$ acts on G through ρ by conjugation. For simplicity, assume that $G = \mathrm{SU}(n)$. Then, there is a rank n complex vector bundle E associated to P, and the flat connection on P induces a $\bar{\partial}$-operator on E. Over a surface, this is always integrable and so, E acquires the structure of a holomorphic bundle. It turns out that with this holomorphic structure, E is *semi-stable*, i.e.,

$$\frac{\deg(F)}{\mathrm{rk}(F)} \leqslant \frac{\deg(E)}{\mathrm{rk}(E)},$$

for all holomorphic subbundles $F \subset E$. The number $\deg(E)/\mathrm{rk}(E)$ is usually denoted by $\mu(E)$ and called the slope of E. When E is an $SU(n)$-bundle, $\mu(E) = 0$, but the notion of stability is important for holomorphic bundles of arbitrary degree. A bundle E is said to be *stable* if strict inequality holds above, for all non-zero proper subbundles of E. Finally, E is said to be *poly-stable* if it is a direct sum of stable bundles.

Algebraic geometers have studied the problem of constructing moduli spaces of vector bundles, and it turns out that it is not possible to give the space of isomorphism classes of all bundles on Σ the structure of a variety. However, Mumford showed that by restricting attention to semi-stable bundles, one can obtain a good moduli space.

There is, therefore, a map from $\mathbf{M}$ to the moduli space of semi-stable bundles of rank n and degree 0, given by taking a flat connection to the semi-stable holomorphic structure induced by it. The famous theorem of Narasimhan and Seshadri [26] states that this map is a homeomorphism.

Higgs bundles were introduced by Hitchin [19] and are of interest for a variety of reasons. One is that they provide the framework for the generalisation of the theorem of Narasimhan and Seshadri to non-compact groups. From the point of view of gauge theory, a Higgs bundle over Σ is a pair (A, Φ), where A is a unitary connection on a C^∞ complex vector bundle E and $\Phi \in \Omega^{1,0}(\Sigma; \mathrm{End}(E))$, satisfying Hitchin's equations. These are a set of elliptic non-linear differential equations for A and Φ, and one consequence is that Φ is holomorphic with respect to the holomorphic structure on E induced by A.

From the holomorphic point of view, a Higgs bundle is a pair (E, Φ), consisting of a holomorphic vector bundle E and a Higgs field

$$\Phi \in H^0(\Sigma; \mathrm{End}(E) \otimes K),$$

where K, as usual, is the canonical bundle on Σ. There is a generalisation of the above notion of stability to Higgs bundles: the condition is the same but it only applies to Φ-invariant subbundles of E. A solution to Hitchin's equations gives rise to a holomorphic Higgs bundle by giving E the holomorphic structure induced by the connection A, and this Higgs bundle turns out to be poly-stable. Moduli spaces of Higgs bundles can be constructed both from the gauge theory point of view (this was done by Hitchin [19]) and from the algebro-geometric point of view (done by Nitsure [27] and Simpson [30]).

The analogue of the theorem of Narasimhan and Seshadri, that the moduli space of solutions to Hitchin's equations is isomorphic to the moduli space of poly-stable Higgs bundles, is true. It was proved by Hitchin [19] in the case of bundles of rank 2 and, in greater generality, by Simpson [28].

Hitchin observed that if (A, Φ) is a solution to his equations, the connection $A+\Phi+\Phi^*$ is a flat $\mathrm{SL}(n, \mathbb{C})$ connection. If B is any $\mathrm{SL}(n, \mathbb{C})$-connection, choosing a Hermitian metric on the bundle allows one to write $B = A+\Phi+\Phi^*$ for a unitary connection A and a Higgs field Φ. It is a theorem of Donaldson [13] and, more generally, Corlette [10], that when B is a flat reductive connection, there exists a metric on E such that the associated pair (A, Φ) satisfies Hitchin's equations. Thus, the moduli space of flat reductive $\mathrm{SL}(n, \mathbb{C})$-connections is isomorphic to the moduli space of poly-stable Higgs bundles of rank n and degree 0.

Other vector bundles with extra structure have been studied extensively. One example is that of the *vortex pairs* of Bradlow [6], which consist of a bundle together with a section. There is a stability condition for these, involving a parameter, and stable pairs correspond to solutions of the vortex equation, which also involves a parameter. The moduli spaces of vortex pairs have been studied carefully by Thaddeus [31]. He shows that the moduli spaces for different values of the parameter are related and, among other things, uses this to calculate their Betti numbers, and to prove the Verlinde formula.

A generalisation of the vortex pairs are the *triples* of Bradlow and García-Prada [9], which consist of a pair of bundles and a map between them. These, and also Higgs bundles, are examples of the *Q-bundles* of Alastair King. We shall not give the definition here, but refer the reader to Section 2.4 for details. There is a stability condition for Q-bundles, which involves a number of parameters. This is a generalisation of the stability condition for vortex pairs and triples.

For certain Q-bundles there is an obvious way of defining an associated Higgs bundle and vice versa. For a special value of the parameters involved in the stability condition for Q-bundles, it is obvious that Higgs bundle stability implies Q-bundle stability. We show that the converse is true for a particularly simple kind of Q-bundle.

In this thesis, we shall use the interpretation of the moduli space of flat connections in terms of Higgs bundles to study the topology of some of these spaces. An important ingredient is the theorem about Q-bundles mentioned above. We have two main results.

The first is a calculation of the Betti numbers of the moduli space of stable Higgs bundles of rank 3 and degree 1 with fixed determinant. This is the space of representations of a universal central extension of $\pi_1\Sigma$, in $\mathrm{SL}(3,\mathbb{C})$. The reason for considering this space, instead of the space of representations of $\pi_1\Sigma$, is that it is smooth, which is necessary for the calculation to work. We follow the Morse theory approach of Hitchin's calculation in [19] in the rank 2 case, using the L^2-norm squared of the Higgs field as a Morse-Bott function. The main new ingredient lies in the determination of the critical submanifolds of the Morse function: this involves Bradlow's vortex pairs mentioned above, and as a result, we see the moduli space of rank 2 vortex pairs, for certain values of the parameter, inside the moduli space of Higgs bundles. The proof of this is an application of the theorem about Q-bundles. In order to carry the calculation through, we use Thaddeus' calculation in [31] of the Betti numbers of the vortex moduli spaces.

The other result is a study of the number of connected components of the moduli space of flat reductive $\mathrm{Sp}(4,\mathbb{R})$-connections on the surface Σ. Even though $\mathrm{Sp}(4,\mathbb{R})$ is a real group, there is an interpretation in terms of Higgs bundles (see Section 2.2 for details). The idea to use Higgs bundles to study flat connections for a real group is due to Hitchin [19] and [21].

The idea is again to use $\|\Phi\|^2$ as a Morse-Bott function. However, the moduli space is not smooth, so this approach does not work directly. For this reason, we use topological invariants of the Higgs bundles to distinguish subspaces of the moduli space, each of which is a union of connected components. It is a consequence of Uhlenbeck's weak compactness theorem that the function $\|\Phi\|^2$ is proper so, if we can show that on each of these subspaces the space of local minima of $\|\Phi\|^2$ is connected, it follows that these subspaces are in fact connected components.

The most obvious of these topological invariants is the first Chern class, d, of the complex vector bundle obtained by a reduction of structure group to the maximal compact subgroup $\mathrm{U}(2) \subset \mathrm{Sp}(4,\mathbb{R})$. Denote the corresponding subspace of the moduli space of flat $\mathrm{Sp}(4,\mathbb{R})$-connections by $\mathcal{M}_d$. From the Higgs bundle point of view, a well-known Milnor-Wood type inequality can easily be proved; this states that $|d| \leqslant 2g-2$. Furthermore, in the extremal case $|d| = 2g-2$, we show that there is an isomorphism between $\mathcal{M}_d$ and the moduli space of poly-stable Higgs bundles of rank 2 with a non-degenerate quadratic form, where the Higgs field is symmetric with respect to the quadratic form and twisted by the square of the canonical bundle. The main point of the proof is to show that the stability conditions coming from the two different points of view are identical. The proof of this is another application of the theorem about Q-bundles.

The Higgs bundles with a quadratic form have structure group $\mathrm{O}(2,\mathbb{C})$ and hence, we get the Stiefel-Whitney classes w_1 and w_2 as further topological invariants, when $|d| = 2g-2$. The case of $w_1 = 0$ is relatively simple to treat but, when $w_1 \neq 0$, we need to use the spectral curve introduced by Hitchin in [20], and the mod 2 index theorem of Atiyah-Singer to identify the local minima of the function $\|\Phi\|^2$ and show that they form connected subspaces.

When $|d| < 2g-2$, we conjecture that the subspaces $\mathcal{M}_d$ are connected but, we are only able to prove this when $d = 0$. We can, however, show that $\mathcal{M}_d \neq \varnothing$, and this gives a lower bound on the number of connected components of $\mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$. We also discuss two possible approaches to a general proof.

This thesis is organised as follows:

In Chapter 2, we collect some general facts. Section 2.1 is an introduction to Higgs bundles. In Section 2.2, we describe how to interpret flat bundles in terms of Higgs bundles, and which Higgs bundles correspond to flat $\mathrm{Sp}(4,\mathbb{R})$-bundles. The Morse theory approach to studying the moduli space of Higgs bundles is reviewed in Section 2.3. Section 2.4 is a brief introduction to Q-bundles, and we prove the theorem about Q-bundles mentioned above.

Chapter 3 contains the calculation of the Betti numbers of the moduli space of Higgs bundles of rank 3 and degree 1. The result is stated in Section 3.1, while the structure of the argument is explained in Section 3.2. Sections 3.3, 3.4, and 3.5 are taken up by the details of the calculation.

In Chapter 4, we study the moduli space of flat reductive $\mathrm{Sp}(2n, \mathbb{R})$-connections from the point of view of Higgs bundles. Section 4.1 contains a proof, using Higgs bundles, of a well-known Milnor-Wood type inequality. In Section 4.2, we are concerned with the main result of the chapter. This is the study of the number of connected components of the moduli space when $n = 2$. Finally, in Section 4.3, we briefly mention a possible application of the aforementioned result to 4-manifolds, which are fibred over a surface.

Chapter 2

General Facts

2.1 Higgs Bundles

Let G^c be a complex semisimple Lie group and let $G_0 \subset G^c$ be the maximal compact subgroup. Denote the Lie algebras of these groups by $\mathfrak{g}^c$ and $\mathfrak{g}_0$, respectively. Thus $\mathfrak{g}^c$ has a compact real structure σ; the $+1$ eigenspace of σ is $\mathfrak{g}_0$, and the -1 eigenspace we denote by $\mathfrak{g}_0^\perp$ (this is the orthogonal complement of $\mathfrak{g}_0$ with respect to the Killing form).

Let Σ be closed Riemann surface of genus $g \geqslant 2$ and let P be a principal G_0-bundle on Σ, then we have the adjoint bundle

$$\operatorname{Ad} P = P \underset{\operatorname{Ad}}{\times} \mathfrak{g}_0.$$

Denote by P^c the G^c-bundle corresponding to P, then $\operatorname{Ad} P^c$ has a real structure induced by σ which, for simplicity, we shall call σ as well. Of course the corresponding real subbundle is just $\operatorname{Ad} P$ (in other words $\operatorname{Ad} P^c = \operatorname{Ad} P \otimes \mathbb{C}$). From the point of view of gauge theory, a G_0-Higgs bundle is a pair (A, Φ) consisting of a connection A on P and a $(1,0)$-form $\Phi \in \Omega^{1,0}(\Sigma; \operatorname{Ad} P^c)$ satisfying Hitchin's equations

$$F(A) - [\Phi, \sigma(\Phi)] = 0 \tag{2.1}$$
$$\bar{\partial}_A \Phi = 0. \tag{2.2}$$

Here $\bar{\partial}_A$ is the $(0,1)$-part of the covariant derivative d_A defined by the connection A.

Now, suppose that $G_0 = \mathrm{SU}(n)$ and $G^c = \mathrm{SL}(n, \mathbb{C})$; we thus have an associated vector bundle $E = P \underset{G}{\times} \mathbb{C}^n$. Let (A, Φ) be a solution to Hitchin's equations; then A defines a holomorphic structure on E through $\bar{\partial}_A$ and equation (2.2) shows that Φ is a holomorphic section of $\operatorname{End}_0(E) \otimes K$. Here

End_0 denotes the trace-free endomorphisms and, as usual, K is the canonical bundle of Σ. From a holomorphic point of view a Higgs bundle can thus be considered as a pair (E, Φ), where E is a holomorphic vector bundle and $\Phi \in H^0(\Sigma; \mathrm{End}_0(E) \otimes K)$.

Note that, from this point of view, there is no need for the Higgs field to be twisted by the canonical bundle K; the concept makes equally good sense for $\Phi \in H^0(\Sigma; \mathrm{End}(E) \otimes L)$, for any holomorphic linebundle L. Furthermore, in the present description the holomorphic determinant bundle of E is $\Lambda^n E = \mathcal{O}$ but, of course, there is no need to specify $\Lambda^n E$ in the definition of a Higgs bundle and, correspondingly, no need to require Φ to be trace-free.

Finally, one can define Higgs bundles from a principal bundle point of view: if G^c is a complex Lie group, a G^c-Higgs bundle is a pair (P^c, Φ), where P^c is a holomorphic principal G^c-bundle and $\Phi \in H^0(\Sigma; \mathrm{Ad}\, P^c \otimes L)$ for some fixed holomorphic linebundle L. When $G^c \subset \mathrm{GL}(n, \mathbb{C})$, a G^c-Higgs bundle gives rise to a Higgs bundle in the vector bundle sense, in general with some extra structure reflecting the definition of G^c—we shall see an example of this with $G^c = \mathrm{Sp}(2n, \mathbb{C})$. In this thesis we shall stick mainly to the vector bundle point of view.

Let us now describe the notion of stability of Higgs bundles, introduced by Hitchin [19]. The slope of a Higgs bundle is defined to be the slope of the underlying vector bundle:

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Furthermore, a subbundle $F \subset E$ is called Φ-*invariant* if $\Phi(F) \subset F \otimes K$; in other words, (F, Φ) is a sub-Higgs bundle of (E, Φ). A Higgs bundle is said to be *stable* if for any non-trivial proper Φ-invariant subbundle F the inequality

$$\mu(F) < \mu(E) \tag{2.3}$$

holds. It is called *poly-stable* if it is a direct sum of stable Higgs bundles. Finally it is called *semi-stable* if equality is allowed in (2.3).

Next we come to the connection between the gauge theoretic and the holomorphic point of view. Let P be a principal bundle with structure group $G_0 = \mathrm{SU}(n)$ and let (A, Φ) be a solution to Hitchin's equations on P. Then a vanishing theorem (see Hitchin [19] and Simpson [28]) states that the corresponding Higgs bundle (E, Φ) is polystable. The theorem of Hitchin and, more generally, Simpson, provides the converse.

Theorem 2.1 (Hitchin [19] and Simpson [28]). *Let (E, Φ) be a poly-stable Higgs bundle. There is a unique Hermitian metric on E such that* *the pair* (A, Φ) *satisfies Hitchin's equations* (2.1) *and* (2.2). *Here,* A *is the unitary connection determined by the Hermitian metric and the holomorphic structure on* E.

Our objects of study are various moduli spaces of Higgs bundles. The above theorem allows these to be considered from the point of view of both gauge theory and algebraic geometry.

We denote by $\mathcal{A}$ the space of connections on P; this is an affine space modeled on $\Omega^1(\Sigma; \operatorname{Ad} P)$. We denote by Ω the space of Higgs fields $\Omega^{1,0}(\Sigma; \operatorname{Ad} P^c)$. Denote the space of solutions to Hitchin's equations by $\mathcal{C} \subset \mathcal{A} \times \Omega$ and let $\mathcal{G} = \Omega^0(P \underset{\operatorname{Ad}}{\times} G_0)$ be the gauge group. From the point of view of gauge theory the moduli space is then $\mathcal{M} = \mathcal{C}/\mathcal{G}$. The usual type of arguments, involving Sobolev completions, shows that around irreducible solutions to the equations $\mathcal{M}$ has the structure of a smooth manifold. However, in general $\mathcal{M}$ will have singularities. For details in the rank 2 case see [19].

From the point of view of algebraic, or holomorphic, geometry, we are considering the space of polystable Higgs bundles (E, Φ) modulo isomorphism. From this point of view, moduli spaces of Higgs bundles have been constructed by Nitsure [27] and in greater generality by Simpson [30]. From Theorem 2.1 it follows that $\mathcal{M}$ and the algebraic geometry moduli space $\mathcal{M}_{\mathrm{alg}}$ are diffeomorphic (actually, the holomorphic structure can also be seen on $\mathcal{M}$ and it is then isomorphic to $\mathcal{M}_{\mathrm{alg}}$). From now on, we shall identify the two and denote them by $\mathcal{M}$.

We shall need the description of the space of infinitesimal deformations $\mathbf{T}$ of a Higgs bundle (P^c, Φ) given by Biswas and Ramanan in [5]. This is the first hypercohomology of the complex of sheaves

$$0 \to \underline{\Omega}^0(\Sigma; \operatorname{Ad} P^c) \xrightarrow{\operatorname{ad}(\Phi)} \underline{\Omega}^{1,0}(\Sigma; \operatorname{Ad} P^c) \to 0,$$

where $\underline{\Omega}^{p,q}(F)$ denotes the sheaf of smooth sections of the bundle of F-valued (p,q)-forms. From this, one easily obtains the 5-term exact sequence, first described by Nitsure [27]:

$$\begin{aligned} H^0(\Sigma; \operatorname{Ad} P^c) \xrightarrow{\alpha} H^0(\Sigma; \operatorname{Ad} P^c \otimes K) \xrightarrow{\beta} \mathbf{T} \\ \xrightarrow{\gamma} H^1(\Sigma; \operatorname{Ad} P^c) \xrightarrow{\delta} H^1(\Sigma; \operatorname{Ad} P^c \otimes K), \end{aligned} \quad (2.4)$$

where α and δ are induced by the map of sheaves

$$\operatorname{ad}(\Phi)\colon \mathcal{O}(\operatorname{Ad} P^c) \to \mathcal{O}(\operatorname{Ad} P^c \otimes K), \quad (2.5)$$

β maps a variation $\dot{\Phi}$ of the Higgs field to $(0, \dot{\Phi})$, and γ maps a variation $(\dot{A}, \dot{\Phi})$ to the variation in the bundle $\dot{A}$. The Higgs bundle (P^c, Φ) corresponds to a smooth point of $\mathcal{M}$ exactly when α is injective, or, equivalently, δ is surjective, and in this case $\mathbf{T}$ is the tangent space to $\mathcal{M}$. The dimension of $\mathbf{T}$ can be calculated from the Riemann-Roch formula.

2.2 Flat Bundles and Higgs Bundles

In this section we give a summary of the connection between flat bundles and Higgs bundles, in order to establish notation, and to have a convenient reference for some key facts.

2.2.1 The Theorem of Donaldson and Corlette

The theorem of Donaldson [13] and, more generally, Corlette [10] provides a way of choosing a preferred metric on a flat bundle. In this subsection we review their results.

Let G be a Lie group, and let $\rho\colon \pi_1\Sigma \to G$ be a representation of the fundamental group of the surface in G. This representation corresponds to a principal G-bundle

$$P = \tilde{\Sigma} \underset{\rho}{\times} G,$$

with a flat connection, which we denote by B. Here $\tilde{\Sigma}$ is the universal cover of Σ, and $\pi_1\Sigma$ acts on G via $\mathrm{Ad} \circ \rho$.

Let $H \subset G$ be a maximal compact subgroup. By a metric on P, we mean a section σ' of the bundle $P/H = \tilde{\Sigma} \times_\rho G/H$, or, equivalently, a $\pi_1\Sigma$-equivariant map

$$\sigma\colon \tilde{\Sigma} \to G/H.$$

To see the reason for this terminology, think of the case $G = \mathrm{GL}(n, \mathbb{C})$ and $H = \mathrm{U}(n)$. In this case, σ' defines a Hermitian metric in the associated vector bundle.

Having a metric on P is equivalent to having a reduction of the structure group of P, from G to H. Thus, a metric defines an inclusion of a principal H-bundle

$$i\colon P_H \hookrightarrow P.$$

The maximal compact subgroup H is characterized as follows: the Killing form, restricted to $\mathfrak{h}$, is negative definite and $\mathfrak{h}$ is maximal with this property. Let $\mathfrak{h}^\perp$ be the complement of $\mathfrak{h}$ with respect to the Killing form, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. The adjoint representation of G splits correspondingly, as the direct sum of $\mathrm{Ad}_{\mathfrak{h}}$ and $\mathrm{Ad}_{\mathfrak{h}^\perp}$. Hence we obtain a splitting of $\mathrm{Ad}\, P$ into the Whitney sum

$$\mathrm{Ad}_{\mathfrak{h}} P \oplus \mathrm{Ad}_{\mathfrak{h}^\perp}.$$

Note that $\mathrm{Ad}_{\mathfrak{h}} P = \mathrm{Ad}\, P_H$. Thinking of the connection B as an element of $\Omega^1(P; \mathfrak{g})$, we can write

$$i^*B = A + \theta,$$

where $A \in \Omega^1(P_H; \mathfrak{h})$ defines a connection in P_H, and where $\theta \in \Omega^1(P_H; \mathfrak{h}^\perp)$ is a tensorial form, and therefore defines an element of $\Omega^1(\Sigma; \mathrm{Ad}_{\mathfrak{h}^\perp} P)$.

The metric σ is called harmonic, when it is an extremum of the energy functional

$$E(\sigma) = \tfrac{1}{2} \int_{\tilde{\Sigma}} |\mathrm{d}\sigma|^2.$$

Note, that $\mathrm{d}\sigma \in \Omega^1(\tilde{\Sigma}; \sigma^* T(G/H))$ and that G/H is a Riemannian manifold so, the above integral makes sense. Furthermore, G/H has a Levi-Civita connection ∇, and we denote by d_∇ the pull-back of this to $\sigma^* T(G/H)$. With this notation, σ is harmonic if and only if

$$\mathrm{d}_\nabla^*(\mathrm{d}\sigma) = 0.$$

In terms of A and θ defined above, this equation is equivalent to

$$\mathrm{d}_A^* \theta = 0, \tag{2.6}$$

and the flatness condition $F(B) = 0$ is equlivalent to the pair of equations

$$F(A) + \tfrac{1}{2}[\theta, \theta] = 0 \tag{2.7}$$
$$\mathrm{d}_A \theta = 0. \tag{2.8}$$

The theorem of Donaldson and Corlette is an existence and uniqueness theorem for harmonic metrics: they will not exist in general but, for certain representations ρ they will. A representation ρ of $\pi_1(\Sigma)$, in a real semi-simple algebraic group G, is said to be *reductive*, if the Zariski closure of $\rho(\pi_1(\Sigma))$ is a reductive subgroup of G.

Theorem 2.2 (Corlette [10] and Donaldson [13]). *Let G be a real semi-simple algebraic group, and let ρ be a representation in G of the fundamental group of a surface Σ. If ρ is reductive there exists a unique harmonic metric in the associated principal G-bundle P over Σ. Conversely, a bundle with a harmonic metric will give rise to a reductive representation of the fundamental group.*

2.2.2 Hitchin's Equations and Simpson's Theorem

We describe how to reinterpret harmonic metrics in terms of solutions to Hitchin's equations. This, in turn, leads to a holomorphic point of view in terms of Higgs bundles.

General Theory

As before let G^c be a complex semisimple Lie group, let $G_0 \subset G^c$ be the maximal compact subgroup, and denote the Lie algebras of these groups by $\mathfrak{g}^c$ and $\mathfrak{g}_0$, respectively. Thus, $\mathfrak{g}^c$ has a compact real structure σ; the $+1$ eigenspace of σ is $\mathfrak{g}_0$, and the -1 eigenspace we denote by $\mathfrak{g}_0^\perp$, this being the orthogonal complement of $\mathfrak{g}_0$ with respect to the Killing form.

Let $P^c \to \Sigma$ be a G^c-bundle with a flat reductive connection B. Apply Theorem 2.2 to get a reduction of the structure group to G_0, and decompose B in a G_0-connection A and a 1-form $\theta \in \Omega^1(\mathrm{Ad}_{\mathfrak{g}_0^\perp} P^c)$. Denoting the tangent space to Σ by T, θ is a section of a bundle with fibre $T \otimes \mathfrak{g}_0^\perp$. Put

$$V^c = T^{c,*} \underset{\mathbb{C}}{\otimes} \mathfrak{g}^c,$$

where T^c is the complexification of T. There is a real structure on V^c, induced from the real structure on T^c and the compact real structure σ on $\mathfrak{g}^c$. By abuse of notation, we shall denote this by σ as well. Denote the real subspace by V and the imaginary subspace by $V^\perp$. Note that V^c has an additional complex structure I, linear with respect to i, which comes from the complex structure on T. Denote the ± 1 eigenspaces of I by $V^{1,0}$ and $V^{0,1}$. Observe, that the following identifications hold:

$$\begin{aligned} V &= T^* \underset{\mathbb{R}}{\otimes} \mathfrak{g}_0 \\ V^\perp &= T^* \underset{\mathbb{R}}{\otimes} \mathfrak{g}_0^\perp \\ V^{1,0} &= T^{(1,0),*} \underset{\mathbb{C}}{\otimes} \mathfrak{g}^c \\ V^{0,1} &= T^{(0,1),*} \underset{\mathbb{C}}{\otimes} \mathfrak{g}^c. \end{aligned}$$

Note also, that the projection onto $V^\perp$, $\Phi \mapsto \Phi - \sigma(\Phi)$, gives an isomorphism $V^{1,0} \cong V^\perp$. Thus, we have an identification $\Omega^{1,0}(\mathrm{Ad}\, P^c) \cong \Omega^1(\mathrm{Ad}_{\mathfrak{g}_0^\perp} P^c)$, and consequently

$$\theta = \Phi - \sigma(\Phi)$$

for a unique $\Phi \in \Omega^{1,0}(\mathrm{Ad}\, P)$. Similarly, we have the usual splitting of the connection A in its $(1,0)$- and $(0,1)$-parts:

$$\mathrm{d}_A = \partial_A + \bar{\partial}_A.$$

The equations (2.6), (2.7), and (2.8) are equivalent to Hitchin's equations (2.1) and (2.2). In this way, we obtain a poly-stable G^c-Higgs bundle from the flat reductive bundle P^c.

Letting $G \subset G^c$ be a real form, $\mathfrak{g}^c$ has a real structure τ whose $+1$ eigenspace is $\mathfrak{g}$, the Lie algebra of G. We want to consider flat G-connections from the point of view of Higgs bundles. Let P be a principal G-bundle with a flat connection B and apply the theorem of Donaldson and Corlette as above to obtain a harmonic metric, and hence a reduction to a bundle P_H with structure group H. Here $H \subset G$ is the maximal compact subgroup, characterized by the Killing form restricted to $\mathfrak{h}$ being negative definite, and by being maximal with respect to this property. Letting P^c be the G^c-bundle obtained from the inclusion $G \hookrightarrow G^c$, we obtain a G^c-Higgs bundle. However, now the connection A will be a connection in P_H, and hence define a holomorphic structure in the complexification P_H^c, and the Higgs field will be a $(1,0)$-form with values in $\mathrm{Ad}_{\mathfrak{h}^{\perp,c}} P_H$. Conversely, a G^c-Higgs bundle, which is of this particular kind, will give rise to a flat G-bundle.

Flat $\mathrm{Sp}(2n,\mathbb{R})$-bundles

We shall consider flat $\mathrm{Sp}(2n,\mathbb{R})$-bundles, and describe the Higgs bundles obtained from them via the above procedure. In order to do this, we need a concrete realization of the pull-back diagram of Lie algebras

$$\begin{array}{ccc} \mathfrak{u}(n) & \longrightarrow & \mathfrak{sp}(2n,\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{sp}(n) & \longrightarrow & \mathfrak{sp}(2n,\mathbb{C}). \end{array}$$

Of course, the corresponding results hold for the Lie groups.

Let $\mathbb{V} = \mathbb{C}^n = \mathbb{R}^{2n}$ be an n-dimensional complex vector space with a Hermitian inner product h. This can be written $h = g + i\omega$, where g is a real inner product, and ω is a symplectic form.

Put $\mathbb{E} = \mathbb{V} \oplus \bar{\mathbb{V}}$, thus the complex structure on $\mathbb{E}$ is $I = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. We can think of $\mathbb{E}$ as a quaternionic vector space, with the complex structure J given by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ defines a real structure σ on $\mathbb{E}$, and there is an isomorphism

$$\gamma\colon \mathrm{End}_{\mathbb{R}}(\mathbb{V}) \xrightarrow{\cong} \mathrm{End}^{\sigma}(\mathbb{E}).$$

Here $\mathrm{End}^{\sigma}(\mathbb{E})$ is the space of endomorphisms which commute with σ. Thus, elements of $\mathrm{End}^{\sigma}(\mathbb{E})$ are real with respect to the real structure on $\mathrm{End}_{\mathbb{C}}(\mathbb{E})$ defined by σ. This real structure is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix}$$

and hence, $\mathrm{End}^{\sigma}(\mathbb{E})$ consists of matrices of the form $\left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$. The map γ is defined as follows: for $A \in \mathrm{End}_{\mathbb{R}}(\mathbb{V})$, write $A = a + b$, where a and b are the complex linear and complex anti-linear parts of A, respectively. Thus $a \in \mathrm{End}_{\mathbb{C}}(\mathbb{V})$ and $b \in \mathrm{Hom}_{\mathbb{C}}(\bar{\mathbb{V}}, \mathbb{V}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{V}, \bar{\mathbb{V}})$. Now define

$$\gamma(A) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

The Hermitian metric h defines an isomorphism $\mathbb{V}^* \cong \bar{\mathbb{V}}$ and, using this, we define a complex symplectic form on $\mathbb{E} \cong \mathbb{V} \oplus \mathbb{V}^*$ by

$$\Omega\big((v, \alpha), (w, \beta)\big) = \alpha(w) - \beta(v).$$

Hence, Ω is given by the matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$, and $\mathfrak{sp}(2n, \mathbb{C})$ consists of those $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ for which $d = -a^T$, $b = b^T$, and $c = c^T$. Here, and throughout, we use the notation A^T exclusively to mean the induced map $\mathbb{W}^* \to \mathbb{V}^*$ for $A \colon \mathbb{V} \to \mathbb{W}$.

Using g to identify $\mathbb{V}$ with $\mathrm{Hom}_{\mathbb{R}}(\mathbb{V}, \mathbb{R})$, we see that $A \in \mathfrak{sp}(2n, \mathbb{R}) \subset \mathrm{End}_{\mathbb{R}}(\mathbb{V})$ if and only if it satisfies $A^T i + iA = 0$. Splitting A in complex linear and complex anti-linear parts a and b, it follows that $a^T = -a$ and $b = b^T$. Hence, the map γ realizes the inclusion $\mathfrak{sp}(2n, \mathbb{R}) \hookrightarrow \mathfrak{sp}(2n, \mathbb{C})$.

The Hermitian metric h on $\mathbb{V}$ defines a Hermitian metric on $\mathbb{E}$, and this realizes $\mathfrak{u}(2n) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{E})$ as those endomorphisms, which satisfy $A = -A^*$, where A^* denotes the adjoint. Finally, use the fact that $\mathrm{Sp}(n)$ is the intersection of $\mathrm{U}(2n)$ and $\mathrm{Sp}(2n, \mathbb{C})$ to obtain $\mathfrak{sp}(n) \hookrightarrow \mathfrak{sp}(2n, \mathbb{C})$.

We need to make one more observation: using the metric g to identify $\mathbb{V}$ with $\mathrm{Hom}_{\mathbb{R}}(\mathbb{V}, \mathbb{R})$, we have $A^T = -A^*$; in other words the transpose of a real matrix corresponds to the conjugate transpose of a complex matrix, under the identification $\mathbb{C} = \mathbb{R}^2$. Hence, $\mathfrak{u}(n) \subset \mathrm{End}_{\mathbb{R}}(\mathbb{V})$ consists of those complex linear endomorphisms, which satisfy $a = -a^*$, and therefore γ gives inclusions $\mathfrak{u}(n) \hookrightarrow \mathfrak{sp}(n)$ and $\mathfrak{u}(n) \hookrightarrow \mathfrak{sp}(2n, \mathbb{R})$.

Let P be a $\mathrm{Sp}(2n, \mathbb{R})$-bundle with a flat reductive connection B. Applying Theorem 2.2, we obtain a poly-stable Higgs bundle (E, Φ). From the preceding discussion we see that this will be of the form

$$E = V \oplus V^* \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \tag{2.9}$$

where $b = b^T$ and $c = c^T$; in other words

$$b \in H^0(S^2 V \otimes K)$$

and

$$c \in H^0(S^2 V^* \otimes K).$$

Conversely, it follows from Simpson's theorem, that any poly-stable Higgs bundle of the form (2.9) gives rise to a flat reductive $\mathrm{Sp}(2n,\mathbb{R})$-bundle. Thus, the space of flat reductive $\mathrm{Sp}(2n,\mathbb{R})$-connections on Σ is homeomorphic to the moduli space of poly-stable Higgs bundles of the type (2.9). We denote this space by $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{R})}$.

To conclude, we describe the infinitesimal deformations of a Higgs bundle of the type (2.9). The 5-term exact sequence (2.4) becomes

$$\begin{aligned} H^0(\Sigma;\mathrm{End}(V)) &\xrightarrow{\alpha} H^0(\Sigma; S^2V\otimes K\oplus S^2V^*\otimes K) \xrightarrow{\beta} \mathbf{T} \\ &\xrightarrow{\gamma} H^1(\Sigma;\mathrm{End}(V)) \xrightarrow{\delta} H^1(\Sigma; S^2V\otimes K\oplus S^2V^*\otimes K). \end{aligned} \tag{2.10}$$

Here, α and δ are induced by the map of sheaves

$$\alpha_1\oplus\alpha_2\colon \mathcal{O}(\mathrm{End}(V))\to\mathcal{O}(S^2V\otimes K)\oplus\mathcal{O}(S^2V^*\otimes K),$$

where

$$\alpha_1(\psi)=\psi b+b\psi^T, \tag{2.11}$$

and

$$\alpha_2(\psi)=-\psi^Tc-c\psi. \tag{2.12}$$

2.3 Morse Theory

2.3.1 The Morse Function

In our analysis of topological properties of the moduli spaces, we shall follow the approach of Hitchin in [19] and use a Morse function which arises as a moment map for an action of the circle on the moduli space $\mathcal{M}$. Here, we give a short review of the method, for more details see [19].

A Kähler metric on $\mathcal{A} \times \Omega$ is defined by

$$g((\psi_1, \Phi_1), (\psi_2, \Phi_2)) = 2i \int_\Sigma \operatorname{tr}(\psi_1^* \psi_2 + \Phi_1 \Phi_2^*),$$

and because it is invariant under the action of the gauge group, it descends to $\mathcal{M}$. The corresponding Kähler form is $\omega_1(X, Y) = g(IX, Y)$. Furthermore, S^1 acts on $\mathcal{M}$ by $(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$, preserving g and the symplectic form ω_1. The function $(A, \Phi) \mapsto -\frac{1}{2}\|\Phi\|^2$ is a moment map for the S^1-action with respect to the symplectic form ω_1. We shall use the negative of this as our Morse function:

$$f(A, \Phi) = \tfrac{1}{2}\|\Phi\|^2. \tag{2.13}$$

Frankel [15, p.5] shows that, in this situation, the function f is a non-degenerate perfect Morse function. Therefore, the Poincaré polynomial of the moduli space $\mathcal{M}$ is given by the Morse counting polynomial

$$P_t(\mathcal{M}) = \sum_N t^{\lambda_N} P_t(N), \tag{2.14}$$

where the sum is over the critical submanifolds, and λ_N is the index of the critical submanifold N, i.e., the dimension of the subbundle of the normal bundle, on which the Hessian of f is negative definite. Thus, our task is to find the critical submanifolds and their indices.

Of course, this approach can only be applied directly when $\mathcal{M}$ is non-singular. In Chapter 3, we shall consider the moduli space of Higgs bundles of rank 3 with rank and degree coprime, which is smooth. In Chapter 4, where we are only interested in finding the connected components of the moduli space, we shall use a slight variation of the argument, which also applies when the moduli space has singular points.

2.3.2 Critical Points and Morse Indices

The analysis of the critical submanifolds rests on the following observation.

Let m be a fixed point for the circle action, represented by (A, Φ). Then, either $\Phi = 0$, or there is a one-parameter family of gauge transformations

$g(\theta)$ such that $g(\theta)(A, \Phi) = (A, e^{i\theta}\Phi)$. In the latter case, let $\psi = \frac{d}{d\theta}g(\theta)|_{\theta=0}$ be the infinitesimal gauge transformation generating this family. We then have

$$d_A\psi = 0 \tag{2.15}$$

$$[\psi, \Phi] = i\Phi. \tag{2.16}$$

Equation (2.15) says that ψ is covariant constant so, taking eigenbundles for ψ, we obtain a decomposition of E as a direct sum of holomorphic subbundles:

$$E = \bigoplus_j U_j, \tag{2.17}$$

where ψ acts on U_j by im_j for some real numbers m_j. Furthermore, when (E, Φ) is stable, it follows from (2.16) that consecutive eigenvalues of ψ differ by i, i.e. $m_{j+1} = m_j + 1$, and that Φ maps U_j to $U_{j+1} \otimes K$ non-trivially. When (E, Φ) is only polystable, it splits as a direct sum of Higgs bundles of this type, all of the same slope.

Conversely, any stable Higgs bundle of this form represents a critical point of f (see [21]).

Next we shall show how to read off the Morse indices at a critical point from the action of the infinitesimal gauge transformation ψ on the tangent space to $\mathcal{M}$ (Hitchin did this in a different way in [21, p. 646f.]). This, in turn, can be easily determined from the exact sequence (2.4).

Let X be the vector field on $\mathcal{M}$ generating the circle action; then

$$\operatorname{grad}(f) = -iX$$

which is exactly the condition for $-f$ to be the moment map. A point $m \in \mathcal{M}$ is a fixed point of the circle action if and only if $X_m = 0$. In this case, we can define an endomorphism H_X of $T_m\mathcal{M}$ by

$$H_X = \nabla(X)_m,$$

where ∇ is any connection on $T\mathcal{M}$. Clearly $-iH_X = H_f$, the Hessian of f. We shall express the eigenvalues of H_X in terms of those of ψ.

Let Y be the vector field on $\mathcal{C}$ generating the circle action there; then Y is given by $Y_{(A,\Phi)} = (0, i\Phi)$. There is also the vector field on $\mathcal{C}$ given by ψ which is $Z = (-d_A\psi, [\psi, \Phi])$. Finally, there is the vector field

$$\tilde{Y} = Y - Z = (d_A\psi, i\Phi - [\psi, \Phi]). \tag{2.18}$$

Note that $\tilde{Y}$ vanishes along the fibre $F = \pi^{-1}(m)$ exactly when X vanishes at m, i.e., when m is a fixed point. Thus, $\tilde{Y}$ defines an endomorphism $H_{\tilde{Y}}$ of $T_{(A,\Phi)}\mathcal{C}$ as above, given by

$$H_{\tilde{Y}}(\dot{A}, \dot{\Phi}) = (-[\psi, \dot{A}], i\dot{\Phi} - [\psi, \dot{\Phi}]). \tag{2.19}$$

Clearly the diagram

$$\begin{array}{ccc} T_{(A,\Phi)}\mathcal{C} & \xrightarrow{H_{\tilde{Y}}} & T_{(A,\Phi)}\mathcal{C} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ T_m\mathcal{M} & \xrightarrow{H_X} & T_m\mathcal{M} \end{array}$$

commutes, and therefore we see from (2.19) that, if ψ acts on $(\dot{A}, \dot{\Phi})$ with eigenvalues (im, in) then H_X acts on their images under π with eigenvalues $(-im, i(1-n))$ and, consequently, the eigenvalues of H_f are $(-m, 1-n)$.

2.4 Q-bundles

The notion of Q-bundle, due to Alastair King, provides a general framework for considering a large number of the variuos kinds of vector bundles with extra structure, which have been studied in recent years. Higgs bundles, the vortex pairs of Bradlow [6], and the triples of Bradlow and García-Prada [9], are all examples of Q-bundles.

2.4.1 Definitions

A quiver is a directed graph, specified by a set of vertices Q_0 and a set of arrows Q_1, together with head and tail maps $h, t\colon Q_1 \to Q_0$. We shall only consider finite quivers, i.e. quivers for which Q_0 and Q_1 are finite.

Definition 2.3. A Q-bundle over a Riemann surface Σ is a collection of holomorphic vector bundles $\{E_i\}_{i\in Q_0}$ over Σ and a collection of holomorphic maps $\{\phi_a\colon E_{t(a)} \to E_{h(a)}\}_{a\in Q_1}$.

A twisted Q-bundle is given by in addition specifying a linebundle L_a for each arrow a. The maps ϕ_a should then go $\phi_a\colon E_{t(a)} \to E_{h(a)} \otimes L_a$.

We shall only consider Q-bundles of a particularly simple form: let Q be a quiver with k vertices and exactly one arrow connecting each pair of vertices in each direction. Let $I = \{1, \dots, k\}$. We denote the vertices by $\{v_i\}_{i\in I}$ and the arrows by $\{a_{ij}\}_{(i,j)\in I^2}$, where a_{ij} is the arrow going from v_i to v_j. We shall consider Q-bundles where all the maps are twisted by a fixed linebundle L. Thus a Q-bundle, twisted by L, is a pair

$$\mathbf{E} = (\underline{E}, \underline{\Phi}),$$

where $\underline{E} = \{E_1, \dots, E_k\}$ and $\underline{\Phi} = \{\phi_{ij}\}$. Here, each E_i is a holomorphic vector bundle on Σ and ϕ_{ij} is a holomorphic section of $\mathrm{Hom}(E_j, E_i \otimes L)$.

There is a stability condition for Q-bundles, which generalises the stability conditions for the bundles with extra structure mentioned above. In our case this condition is as follows. It depends on k parameters $\alpha = \{\alpha_1, \dots, \alpha_k\}$. The α-slope of a Q-bundle $\mathbf{E}$ is by definition

$$\mu_\alpha(\mathbf{E}) = \frac{\sum(\alpha_i \operatorname{rk}(E_i) + \deg(E_i))}{\sum \operatorname{rk}(E_i)}.$$

Note that this only depends on the topological types of the bundles E_i. A Q-bundle is *α-semi-stable* if $\mu_\alpha(\mathbf{F}) \leqslant \mu_\alpha(\mathbf{E})$ for any Q-subbundle $\mathbf{F}$. Furthermore, $\mathbf{E}$ is *α-stable* if we have strict inequality above whenever $\mathbf{F}$ is non-zero and proper. We shall only have use for the case when all the α_i are zero, and we shall, therefore, use the word *stable* rather than α-stable. In this case, the α-slope is simply

$$\mu_\alpha(\mathbf{E}) = \mu\left(\bigoplus E_i\right). \tag{2.20}$$

2.4.2 The Higgs Bundle Associated to a Q-bundle

Given a Q-bundle $\mathbf{E} = (\underline{E}, \underline{\Phi})$, we can define an associated Higgs bundle (E, Φ) by putting

$$E = \bigoplus_{i\in I} E_i \quad \text{and} \quad \Phi = (\phi_{ij}), \tag{2.21}$$

where (ϕ_{ij}) is the matrix of Φ with respect to the above direct sum decomposition of E. Note that (2.20) says that

$$\mu_\alpha(\mathbf{E}) = \mu(E). \tag{2.22}$$

Because of this we shall from now on denote the α-slope simply by $\mu(\mathbf{E})$, and call it the slope.

Let $\mathbf{F}$ be a Q-subbundle of $\mathbf{E}$. Then, obviously, $F \subset E$ is a Φ-invariant subbundle. Conversely, if a Φ-invariant subbundle $F \subset E$ is of the form $F = \bigoplus F_i$ for subbundles $F_i \subset E_i$, then the collection $\underline{F} = \{F_i\}$ defines a Q-subbundle $\mathbf{F}$ of $\mathbf{E}$. From the first of these observations and (2.22), we see that (semi-)stability of (E, Φ) implies (semi-)stability of $\mathbf{E}$. The following theorem says that the converse is true when the Q-bundle $\mathbf{E}$ is of a particularly simple form. It would be interesting to investigate whether this holds for more general $\mathbf{E}$.

Theorem 2.4. *Let Q be a quiver with two vertices and one arrow connecting the vertices in each direction, and let $\mathbf{E} = (\{E_1, E_2\}, \{\phi_{12}, \phi_{21}\})$ be a Q-bundle. Let (E, Φ) be the associated Higgs bundle as above; thus $E = E_1 \oplus E_2$ and*

$$\Phi = \begin{pmatrix} 0 & \phi_{12} \\ \phi_{21} & 0 \end{pmatrix}.$$

Then $\mathbf{E}$ is (semi-)stable if and only if (E, Φ) is.

Proof. The bundle L by which we are twisting is completely irrelevant to the argument so, for ease of notation, we shall assume that $L = \mathcal{O}$. Also, the following proof is the case of $\mathbf{E}$ being semi-stable; we only have to replace "$\leqslant$" by "$<$" in (2.25) to obtain the proof in the case of $\mathbf{E}$ being stable.

Assume that $\mathbf{E}$ is semi-stable. We need to show that $\mu(F') \leqslant \mu(E)$ for any Φ-invariant subbundle $F' \subset E$. Let $\pi_i \colon E \to E_i$ be the projection on the ith factor. Let $F_i \subset E_i$ and $G_i \subset F'$ be the subbundles which are generically the image and kernel of π_i, respectively. Then F_1 and G_2 are contained in E_1, F_2 and G_1 are contained in E_2, and we have sequences of vector bundles

$$0 \to G_i \to F' \to F_i \to 0,$$

which are generically short exact. Hence,

$$\deg(F') \leqslant \deg(G_i) + \deg(F_i).$$

Putting $F = F_1 \oplus F_2$ and $G = G_1 \oplus G_2$, it follows that

$$2\deg(F') \leqslant \deg(F) + \deg(G) \tag{2.23}$$

and, obviously,

$$2\,\mathrm{rk}(F') = \mathrm{rk}(F) + \mathrm{rk}(G). \tag{2.24}$$

We claim that F and G are Φ-invariant and, therefore, define Q-subbundles of $\mathbf{E}$. First, let $x_1 \in F_1$. If we write $x_1 = \pi_1(x)$ for some $x = x_1 + x_2$ in F', then

$$\Phi(x) = \Phi(x_1) + \Phi(x_2).$$

By our assumption on the matrix for Φ, it follows that $\Phi(x_1) \in E_2$ and $\Phi(x_2) \in E_1$. Then $\pi_1(\Phi(x)) = \Phi(x_2) \in E_1$ and $\pi_2(\Phi(x)) = \Phi(x_1) \in E_2$. But $\Phi(x) \in F'$ because F' is Φ-invariant, and thus $\Phi(x_2) \in F_1$ and $\Phi(x_1) \in F_2$. Of course, we can repeat the argument with $x_2 \in F_2$ and hence, F is Φ-invariant.

The proof that G is Φ-invariant is similar. Let $x_1 \in G_2$. By assumption, $\Phi(x_1) \in E_2$. But $G_2 \subset F'$, so $\Phi(x_1) \in F'$ as well. It follows that $\Phi(x_1) \in G_1$ and thus, G is Φ-invariant.

We have thus seen that F and G define Q-subbundles of $\mathbf{E}$ and from semistability of $\mathbf{E}$, we get

$$\deg(F) \leqslant \frac{\mathrm{rk}(F)}{\mathrm{rk}(E)}\deg(E) \quad \text{and} \quad \deg(G) \leqslant \frac{\mathrm{rk}(G)}{\mathrm{rk}(E)}\deg(E). \tag{2.25}$$

Finally, combining (2.23), (2.24), and (2.25), we see that

$$\begin{aligned} 2\deg(F') &\leqslant \frac{\mathrm{rk}(F) + \mathrm{rk}(G)}{\mathrm{rk}(E)}\deg(E) \\ &= \frac{2\,\mathrm{rk}(F')}{\mathrm{rk}(E)}\deg(E). \end{aligned}$$

Thus, $\mu(F') \leqslant \mu(E)$ and the proof is finished. $\square$

Chapter 3

The Moduli Space of Rank 3 Higgs Bundles

3.1 Statement of the Result

In this chapter, our aim is to calculate the Betti numbers of the moduli space of rank 3 Higgs bundles on a smooth closed Riemann surface of genus $g \geqslant 2$. In order to have a smooth moduli space, we shall restrict attention to bundles which have rank and degree coprime. Thus we let $\mathcal{M}$ be the moduli space of stable rank 3 Higgs bundles with fixed determinant bundle Λ_0 of degree d coprime to 3. This space is smooth (see [19]). In this case, the exact sequence (2.4) becomes

$$\begin{aligned} H^0(\Sigma; \mathrm{End}_0(E)) \overset{\alpha}{\to} H^0(\Sigma; \mathrm{End}_0(E) \otimes K) \overset{\beta}{\to} \mathbf{T} \\ \overset{\gamma}{\to} H^1(\Sigma; \mathrm{End}_0(E)) \overset{\delta}{\to} H^1(\Sigma; \mathrm{End}_0(E) \otimes K). \end{aligned} \tag{3.1}$$

The map α is always injective and δ is surjective.

From the point of view of gauge theory, we are considering gauge equivalence classes of solutions (A, Φ) to Hitchin's equations (2.1) and (2.2) which, in the present context, take the form

$$\begin{aligned} F(A)^{\perp} + [\Phi, \Phi^*] &= 0 \\ \bar{\partial}_A \Phi &= 0. \end{aligned}$$

Here A is a unitary connection on a rank 3 vector bundle with a fixed induced connection A_0 on the determinant bundle, and Φ is a $(0,1)$-form with values in the traceless endomorphisms. $F(A)^{\perp}$ denotes the traceless part of the curvature.

In terms of representations of the fundamental group, we are considering representation of a central extension (cf. [19]): there is a universal central extension Γ of $\pi_1\Sigma$, generated by elements $A_1, B_1, \dots, A_g, B_g$ and a central element C subject to the relation $\prod[A_i, B_i] = C$, and $\mathcal{M}$ is the moduli space of irreducible representations of Γ in $SL(3,\mathbb{C})$ which take C to a fixed non-trivial central element determined by the first Chern class of the bundle. Thus, our calculation gives the Betti numbers of this purely topologically defined space.

We shall also state the result for the moduli space $\mathcal{M}'$ where the determinant bundle (of degree d coprime to 3) is allowed to vary. This calculation is actually slightly less involved than the one we shall present.

Theorem 3.1. *Let Σ be a closed Riemann surface of genus $g \geqslant 2$, and let Λ_0 be a holomorphic line bundle on Σ of degree d with $(d,3) = 1$. Let $\mathcal{M}$ be the moduli space of rank 3 stable Higgs bundles on Σ with fixed determinant bundle Λ_0. The Poincaré polynomial of $\mathcal{M}$ is*

$$\begin{aligned}
P_t(\mathcal{M}) =& \frac{(1+t)^{4g-4}}{(1-t)^4}\Big(2t^2 + t^4 + 2t^{2g} + 2t^{2g+2} - \tfrac{1}{4}t^{4g-4} - 3gt^{4g-3} \\
&+ (6g^2 + 2g - 3)t^{4g-2} + (11g - 12g^2)t^{4g-1} \\
&+ (6g^2 - 10g + \tfrac{17}{4})t^{4g} - t^{8g-6} - t^{10g-8}\Big) \\
&+ \frac{t^{2g}(1+t)^{2g-4}}{(1-t)^4(1+t^2)^2}\Big(t^{6g-8}(1+t^3)^{2g}(-2g - t^2 + (2g-2)t^4) \\
&+ (1+t)^{2g}(-2t^4 - 2t^6 + t^{2g-4} + 2t^{2g-2} + t^{2g} - t^{4g-2})\Big) \\
&- \frac{2^{2g}t^{2g}(1+t)^{2g-1}}{(1-t)^4} + \frac{2gt^{8g-8}(1+t)^{2g-3}(1+t^3)^{2g-1}}{(1-t)^3(1+t^2)} \\
&+ \frac{2^{2g-1}t^{10g-8}(1+t)^{2g}}{(1-t)^3(1-t^3)} + \frac{t^{4g-4}(1-t)^{2g-1}(1+t)^{2g-1}}{4(1+t^2)} \\
&+ \frac{t^{6g-2}(1+t)^{4g-3}(1+t^2+t^4)}{(t-1)^3(1+t^2)^2(t^6-1)} + \frac{(1+t^5)^{2g}(1+t^3)^{2g-1}}{(t^2-1)(t^4-1)^2(t^3-1)} \\
&+ t^{4g-4}\big((3^{2g}-1)(1+t)^{4g-4} - 3^{2g}\big).
\end{aligned}$$

Let $\mathcal{M}'$ denote the moduli space of stable Higgs bundles of rank 3 *and degree d with $(d,3) = 1$ and any determinant. Then,*

$$P_t(\mathcal{M}') = (1+t)^{2g}\big(P_t(\mathcal{M}) - (3^{2g}-1)t^{4g-4}((1+t)^{4g-4} - 1)\big).$$

Remark 3.2. It is interesting to note that the Poincaré polynomial of $\mathcal{M}'$ does not split as the product of those of the Jacobian and $\mathcal{M}$. This is in contrast to the case of stable bundles (without Higgs field), see [3]. In particular, it follows that tensoring by a linebundle gives a nontrivial action of the group

$$\Gamma_3 = \{L \in \mathrm{Jac}^0(\Sigma) \mid L^3 = \mathcal{O}\} \cong (\mathbb{Z}/3)^{2g}$$

on $H^*(\mathcal{M};\mathbb{Q})$.

Remark 3.3. Some simpler results can be obtained from the formulas of Theorem 3.1. Setting $t = -1$, we see that $\chi(\mathcal{M}) = -3^{2g}$, while $\chi(\mathcal{M}') = 0$. And for a Riemann surface of genus 2, the Poincaré polynomial of $\mathcal{M}$ is

$$\begin{aligned} P_t(\mathcal{M}) =& 1 + 3t^2 + 20t^3 + 54t^4 + 416t^5 + 572t^6 + 376t^7 + 117t^8 \\ &+ 32t^9 + 47t^{10} + 56t^{11} + 42t^{12} + 28t^{13} + 16t^{14} + 8t^{15} + 3t^{16}. \end{aligned}$$

3.2 Strategy of Proof

The proof of Theorem 3.1 follows Hitchin's Morse theory approach as explained in Section 2.3, using the Morse function

$$f(A,\Phi) = \tfrac{1}{2}\|\Phi\|^2.$$

In this section, we outline how it applies in the case of rank 3 Higgs bundles.

The critical points of f corresponding to the absolute minimum $f = 0$ can be easily dealt with: in this case the Higgs field Φ must vanish and conversely, any Higgs bundle with $\Phi = 0$ is fixed by the circle action. Thus, the corresponding critical submanifold, N_0, is the moduli space of stable bundles of rank 3. The index of N_0 is, of course, $\lambda_{N_0} = 0$, and Desale and Ramanan calculated the Poincaré polynomial of N_0 in [11, p.241]. Their formula is

$$\begin{aligned} P_t(N_0) =& (t^2-1)^{-1}(t^4-1)^{-2}(t^6-1)^{-1}\big((t^5+1)^{2g}(t^3+1)^{2g} \\ &- (t^2+1)^2 t^{4g-2}(1+t)^{2g}(1+t^3)^{2g} + (1+t^2+t^4)t^{6g-2}(1+t)^{4g}\big). \end{aligned}$$

From Section 2.3, we know that a Higgs bundle representing a critical point is of the form (2.17). When $\Phi \neq 0$, there must be at least two non-trivial summands in the direct sum decomposition of E. Thus, we need to consider three distinct types of critical points: we shall say that a Higgs bundle (E,Φ) (or the critical point it represents) is of type $(1,2)$ if it is of the form

$$E = L \oplus V, \tag{3.2}$$

where $\mathrm{rk}(L) = 1$ and $\mathrm{rk}(V) = 2$, and where

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix},$$

with $\phi\colon L \to V \otimes K$. Similarly, we say that (E,Φ) is of type $(2,1)$ if it is of the form

$$E = V \oplus L, \tag{3.3}$$

with $\mathrm{rk}(V) = 2$ and $\mathrm{rk}(L) = 1$, and where

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix},$$

with $\phi\colon V \to LK$. Note that in both cases stability of (E,Φ) forces ϕ to be non-zero.

Finally, (E, Φ) is said to be of type $(1, 1, 1)$, if it is of the form

$$E = L_1 \oplus L_2 \oplus L_3, \tag{3.4}$$

with

$$\Phi = \begin{pmatrix} 0 & 0 & 0 \\ \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \end{pmatrix},$$

where $\phi_1 \colon L_1 \to L_2K$ and $\phi_2 \colon L_2 \to L_3K$. Again, ϕ_1 and ϕ_2 must be non-zero because (E, Φ) is stable.

In Section 3.3, we describe the critical submanifolds corrseponding to Higgs bundles of type $(1, 2)$ and $(2, 1)$, the result being given in propositions 3.7 and 3.9. The critical submanifolds corresponding to Higgs bundles of type $(1, 1, 1)$ are described in Section 3.4, with the result being given in Proposition 3.12. The indices of the critical submanifolds are given in Proposition 3.14. From these results, an elementary calculation of the type in [19, pp. 98–99] gives the formula of Theorem 3.1. This calculation is rather long, and was done with the help of the computer programme *Maple*. We omit the details.

3.3 Critical Points of Type $(1,2)$ and $(2,1)$

Let (E,Φ) represent a critical point of f of type $(1,2)$, hence of the form (3.2). Set $d=\deg(E)$, $l=\deg(L)$, and $v=\deg(V)$, so that $d=l+v$. An easy calculation using the Higgs bundle equation $F(A)^{\perp}+[\Phi,\Phi^*]=0$ and the fact that $F(A)=F(A)^{\perp}+\frac{1}{3}F(A_0)\cdot I$, shows that the (critical) value of f at the point represented by (E,Φ) is

$$f=2\pi(l-\tfrac{1}{3}d). \tag{3.5}$$

The fact that (E,Φ) is a stable Higgs bundle, allows us to bound l, and therefore the critical values of f.

Proposition 3.4. *The degree l of the linebundle L satisfies the inequalities*

$$\tfrac{1}{3}d<l<\tfrac{1}{3}d+g-1.$$

Proof. There is a rank 1 subbundle $L'\subset V$, defined by the requirement that $\phi\colon L\to L'K$. Thus ϕ is a non-zero section of the line bundle $L^{-1}L'K$ and hence, $\deg(L^{-1}L'K)\geqslant 0$. There are three obviously Φ-invariant subbundles of E, namely L', $L\oplus L'$, and V. Applying the stability condition to these and combining with the previous inequality gives the stated result. □

It follows that we can construct any Higgs bundle representing a critical point of type $(1,2)$ as follows. First, we choose a holomorphic linebundle L of degree l with $\frac{1}{3}d<l<\frac{1}{3}d+g-1$. Then, we choose a rank 2 bundle V and a non-zero section $\phi\in H^0(\Sigma;L^{-1}VK)$ such that $\Lambda_0=\Lambda^3(L\oplus V)$, and set $E=L\oplus V$ and $\Phi=\left(\begin{smallmatrix}0&0\\ \phi&0\end{smallmatrix}\right)$. But not any V and ϕ will do; they have to be chosen such that (E,Φ) becomes a *stable* Higgs bundle. An application of Theorem 2.4 will show that the condition on V and ϕ is essentially Bradlow's condition of τ-stability (see [6] and [7]). In the case of bundles of rank 2 on a Riemann surface, it takes the following form (cf. Thaddeus [31]).

Definition 3.5. Let σ be a positive rational number. A pair $(\tilde{V},\phi)$ consisting of a holomorphic bundle $\tilde{V}\to\Sigma$ and a non-zero section $\phi\in H^0(\Sigma;\tilde{V})$ is said to be *σ-semi-stable* if for any line bundle $\tilde{U}\subset\tilde{V}$

$$\deg(\tilde{U})\leqslant\tfrac{1}{2}\deg(\tilde{V})-\sigma\quad\text{if }\phi\in H^0(\Sigma;\tilde{U}),$$

and

$$\deg(\tilde{U})\leqslant\tfrac{1}{2}\deg(\tilde{V})+\sigma\quad\text{if }\phi\notin H^0(\Sigma;\tilde{U}).$$

If we have strict inequalities above $(\tilde{V},\phi)$ is said to be *σ-stable.*

Let $r = \deg(\tilde{V})$. For $\sigma \not\equiv \frac{1}{2}r \pmod{\mathbb{Z}}$ and $\sigma < \frac{1}{2}r$, smooth moduli spaces of σ-stable pairs can be constructed; in [31], Thaddeus constructed a moduli space $\mathcal{N}(\sigma, \Lambda)$ of pairs with fixed determinant bundle Λ by geometric invariant theory, and in [7], Bradlow and Daskalopoulos constructed a moduli space $\mathcal{N}(\sigma)$ of pairs with any determinant bundle (of degree r).

Lemma 3.6. *The Higgs bundle (E, Φ) is stable if and only if the pair $(\tilde{V}, \phi)$ is σ-stable for the value*

$$\sigma = -\tfrac{1}{6}d + \tfrac{1}{2}l.$$

Proof. Let Q be the quiver with two vertices and just one arrow between them. Then, $\mathbf{E} = (\{L, V\}, \{\phi\})$ is a Q-bundle twisted by K and (E, Φ) is the Higgs bundle associated to $\mathbf{E}$, as in Section 2.4. The quiver Q satisfies the hypothesis of Theorem 2.4 so, we know that (E, Φ) is stable if and only if the $\mathbf{E}$ is stable. But, it is a well known and easy fact that this is equivalent to the pair $(\tilde{V}, \phi)$ being σ-stable for the value of σ given above.

For completeness, we shall repeat the argument here. Let $L' \subset V$ be defined by $\phi\colon L \to L'K$. There are two types of non-trivial Q-subbundles of $\mathbf{E}$. The first type is of the form $\mathbf{F} = (\{0, U\}, \{0\})$, where $U \subset V$ is a linebundle. Such Q-subbundles are in 1–1 correspondence with linebundles $\tilde{U} = L^{-1}UK \subset \tilde{V}$. The stability condition for these Q-subbundles is

$$\deg(U) < \tfrac{1}{3}d,$$

or, equivalently,

$$\deg(\tilde{U}) < \tfrac{1}{3}d - l + 2g - 2. \tag{3.6}$$

The second type of Q-subbundle, of which there is just one example, is $\mathbf{F}' = (\{L, L'\}, \{\phi\})$. This Q-subbundle corresponds to the subbundle $\tilde{U}' \subset \tilde{V}$ defined by $\phi \in H^0(\Sigma; \tilde{U}')$. The stability condition for $\mathbf{F}'$ is

$$\deg(L) + \deg(L') < \tfrac{2}{3}d,$$

or, equivalently,

$$\deg(\tilde{U}') < \tfrac{2}{3}d - 2l + 2g - 2. \tag{3.7}$$

Finally, note that $\deg(\tilde{V}) = d - 2l + 4g - 4$. Hence, putting $\sigma = -\frac{1}{6}d + \frac{1}{2}l$, we see that the inequality (3.6) corresponds to the second condition in Definition 3.5, while the inequality (3.7) corresponds to the first. $\square$

We can now determine the critical submanifolds.

Proposition 3.7. *The critical submanifolds of f corresponding to critical points of type $(1,2)$ are indexed by integers l satisfying $\frac{1}{3}d < l < \frac{1}{3}d + g - 1$. For each such l, the critical submanifold N_l is given by the pull-back diagram*

$$\begin{array}{ccc} N_l & \xrightarrow{h} & \mathrm{Jac}^l(\Sigma) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi'} \\ \mathcal{N}(\sigma) & \xrightarrow{\det} & \mathrm{Jac}^r(\Sigma) \end{array}$$

where $\sigma = -\frac{1}{6}d + \frac{1}{2}l$ and $r = d - 3l + 4g - 4$. Furthermore, the map π' is the 3^{2g}-fold covering of the Jacobian given by $L \mapsto L^{-3}\Lambda_0 K^2$, the map h is given by $(L \oplus V, \Phi) \mapsto L$, and the map π is given by $(L \oplus V, \Phi) \mapsto (L^{-1}VK, \phi)$, where $\Phi = \left(\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right)$.

Proof. It follows from Lemma 3.6 that the map $\pi\colon \mathrm{Jac}^l(\Sigma) \to \mathrm{Jac}^r(\Sigma)$ is well defined and surjective. Note that $\Lambda^2(\tilde{V}) = \Lambda_0 L^{-3}K^2$. Thus, $(\tilde{V}, \phi)$ determines $(L \oplus V, \phi)$ up to choosing a third root of $\Lambda_0 L^{-3} K^2$ and we obtain the pull-back diagram above. $\square$

Finally, we need to calculate the Poincaré polynomial of the critical submanifold N_l. This is done by slightly adapting the calculation of the Poincaré polynomial of $\mathcal{N}(\sigma)$ of Thaddeus [31, (4.1)]. We briefly recall the version for σ-stable pairs without fixed determinant of [8]. Let i be an integer in the interval $[0, (r-1)/2]$, then the σ-stability condition is the same for all $\sigma \in (\max(0, r/2 - i - 1), r/2 - i)$. Put $\mathcal{N}_i = \mathcal{N}(\sigma)$ for σ in this interval. There are subvarieties $\mathbb{P}W_i^+$ of $\mathcal{N}_i$ and $\mathbb{P}W_i^-$ of $\mathcal{N}_{i-1}$ such that, when these are blown up, we obtain the same variety $\tilde{\mathcal{N}}_i$. Furthermore, $\mathbb{P}W_i^+$ is a $\mathbb{P}^{r+g-2-2i}$-bundle over $S^i\Sigma \times \mathrm{Jac}^{r-i}(\Sigma)$ and $\mathbb{P}W_i^-$ is a $\mathbb{P}^{i-1}$-bundle over $S^i\Sigma \times \mathrm{Jac}^{r-i}(\Sigma)$. Also, if the projection

$$\pi\colon \mathbb{P}W_i^+ \to S^i\Sigma \times \mathrm{Jac}^{r-i}(\Sigma)$$

is composed with the map

$$\begin{aligned} S^i\Sigma \times \mathrm{Jac}^{r-i}(\Sigma) &\to \mathrm{Jac}^i(\Sigma) \\ (D, L) &\mapsto [D] \otimes L \end{aligned}$$

we get the determinant map (and similarly for $\mathbb{P}W_i^-$).

Proposition 3.8. *The Poincaré polynomial of N_l is*

$$P_t(N_l) = \frac{(1+t)^{2g}}{1-t^2} \operatorname*{Coeff}_{x^i} \left(\frac{t^{2d-6l+10g-10-4i}}{xt^4 - 1} - \frac{t^{2i+2}}{x - t^2} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right),$$

where $i = [\frac{2}{3}d] - 2l + 2g - 2$.

Proof. Note that $i = [\frac{2}{3}d] - 2l + 2g - 2$ corresponds to $\sigma = -\frac{1}{6}d + \frac{1}{2}l$. Thus, using Proposition 3.7, N_l fits into the pull-back diagram

$$\begin{array}{ccc} N_l & \longrightarrow & \mathrm{Jac}^l(\Sigma) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{N}_i & \longrightarrow & \mathrm{Jac}^r(\Sigma). \end{array}$$

We can similarly pull back the subvariety $\mathbb{P}W_i^+$ of $\mathcal{N}_i$ to obtain a subvariety $(\mathbb{P}W_i^+)'$ of N_l. Blowing up this, we get a new variety $\tilde{N}_l$. But, this is the same variety as the one obtained by pulling back the blow-up $\tilde{\mathcal{N}}_i$, because they are locally isomorphic, and blowing up does not change the fundamental group. Note also that $(\mathbb{P}W_i^+)'$ is a $\mathbb{P}^{r+g-2-2i}$-bundle over the pull-back of $\mathrm{Jac}^l(\Sigma) \to \mathrm{Jac}^r(\Sigma)$ under the map

$$\begin{aligned} S^i\Sigma \times \mathrm{Jac}^{r-i}(\Sigma) &\to \mathrm{Jac}^r(\Sigma) \\ (D, L) &\mapsto [D] \otimes L. \end{aligned}$$

It is, however, easy to see that this pull-back is isomorphic to $S^i\Sigma \times \mathrm{Jac}^{r-i}(\Sigma)$. Of course, similar remarks apply to $\mathbb{P}W_i^-$. Finally, we make the observation that for $l = \frac{1}{2}[\frac{2}{3}d] + g - 1$ (corresponding to $i = 0$) N_l is a $\mathbb{P}^{r+g-2}$-bundle over $\mathrm{Jac}^r(\Sigma)$ and hence, its Poincaré polynomial is the product of those of the Jacobian and $\mathbb{P}^{r+g-2}$. Altogether, this information allows us to replicate the argument of [31, (4.1)] and arrive at the formula stated. It should be emphasized that the basic reason why the calculation works and no further information about the various projective bundles is needed, is that the Poincaré polynomial of any projective bundle is the product of that of the base and that of the fibre. $\square$

The description of critical submanifolds of type (2,1) is completely analogous. Alternatively, one can note that $E \mapsto E^*$ takes stable Higgs bundles of type (1,2) to stable Higgs bundles of type (2,1). In any case, we have the following

Proposition 3.9. *The critical submanifolds of f corresponding to critical points of type $(2,1)$ are indexed by integers l satisfying $\frac{1}{3}d + 1 - g < l < \frac{1}{3}d$. For each such l, the critical submanifold N_l is given by the pull-back diagram*

$$\begin{array}{ccc} N_l & \xrightarrow{h} & \mathrm{Jac}^l(\Sigma) \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi'} \\ \mathcal{N}(\sigma) & \xrightarrow{\det} & \mathrm{Jac}^r(\Sigma) \end{array}$$

where $\sigma = \frac{1}{6}d - \frac{1}{2}l$ *and* $r = 3l - d + 4g - 4$. *Furthermore, the map* π' *is the* 3^{2g}*-fold covering of the Jacobian given by* $L \mapsto L^3\Lambda_0^{-1}K^2$, *the map* h *is given by* $(V \oplus L, \Phi) \mapsto L$, *and the map* π *is given by* $(V \oplus L, \Phi) \mapsto (LV^*K, \phi)$, *where* $\Phi = \left(\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right)$.

The Poincaré polynomial of N_l *is given by*

$$P_t(N_l) = \frac{(1+t)^{2g}}{1-t^2} \operatorname*{Coeff}_{x^i} \left(\frac{t^{6l-2d+10g-10-4i}}{xt^4-1} - \frac{t^{2i+2}}{x-t^2} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right),$$

where $i = [-\frac{2}{3}d] + 2l + 2g - 2$.

3.4 Critical Points of Type (1,1,1)

Let (E, Φ) be a stable Higgs bundle representing a critical point of f of type $(1,1,1)$, therefore of the form (3.4). It will be convenient to introduce the linebundles

$$\begin{aligned} M_1 &= L_1^{-1} L_2 K, \\ M_2 &= L_2^{-1} L_3 K, \end{aligned}$$

so that $\phi_1 \in H^0(\Sigma; M_1)$ and $\phi_2 \in H^0(\Sigma; M_2)$. Set $l_i = \deg(L_i)$, $m_i = \deg(M_i)$, and $m = m_1 + m_2$, so that $d = l_1 + l_2 + l_3$,

$$m_1 = l_2 - l_1 + 2g - 2 \tag{3.8}$$

and,

$$m_2 = l_3 - l_2 + 2g - 2. \tag{3.9}$$

The value of f at a critical point can be calculated to be

$$f = 2\pi(4g - 4 - m).$$

Again, we obtain some easy numerical bounds.

Proposition 3.10. *The degrees m_1 and m_2 of M_1 and M_2 satisfy the inequalities*

$$\begin{aligned} &m_i \geqslant 0 \quad \textit{for } i = 1, 2 \\ &m_1 + 2m_2 < 6g - 6 \\ &2m_1 + m_2 < 6g - 6. \end{aligned}$$

Note in particular that $0 \leqslant m < 4g - 4$.

Proof. Clearly, the first inequality holds because M_1 and M_2 have non-zero global sections. To prove the other two inequalities, consider the Φ-invariant subbundles $L_2 \oplus L_3$ and L_3. By stability, we get

$$\begin{aligned} l_3 &< \tfrac{1}{3}d, \\ l_2 + l_3 &< \tfrac{2}{3}d. \end{aligned}$$

Using (3.8) and (3.9), we see that these are equivalent to the two last inequalities in the statement of the proposition . $\square$

Again, we may ask whether any Higgs bundle of this form is stable and in this instance, the answer is affirmative.

Lemma 3.11. *Any Higgs bundle (E, Φ) of the form (3.4), such that the bounds of Proposition 3.10 are satisfied, is stable.*

Proof. First, note that the bounds of Proposition 3.10 imply

$$l_3 < \tfrac{1}{3}d \tag{3.10}$$

and

$$l_2 + l_3 < \tfrac{2}{3}d. \tag{3.11}$$

Denote the projections $E \to L_i$ by π_i for $i = 1, 2, 3$. Let $U \subset E$ be any Φ-invariant rank 1 subbundle of E. Then, $\pi_{1|U}$ and $\pi_{2|U}$ cannot both be zero, because then $U = L_1$, which is obviously not Φ-invariant. Similarly, $\pi_{2|U} \neq 0$ implies $\pi_{3|U} \neq 0$. Hence, $\pi \colon U \to L_3$ is always non-zero, and so $\deg(U) < l_3 < \frac{1}{3}d$ by (3.10). Next, let $U \subset E$ be a Φ-invariant rank 2 subbundle. If $U = L_2 \oplus L_3$, (3.11) shows that the stability condition is satisfied. In fact, this is the only Φ-invariant rank 2 subbundle of E. To see this, assume that $\pi_{1|U} \neq 0$. Then $(\pi_2 + \pi_3)_{|U} \colon U \to L_2 \oplus L_3$ is generically, and hence identically of rank 1. It follows that $U = L_1 \oplus U'$, where $U' = (\pi_2 + \pi_3)(U) \subset L_2 \oplus L_3$. By Φ-invariance of U, we get

$$\Phi(L_1) \subset U'K \cap L_2K.$$

As $\Phi(L_1) \neq 0$, it follows that $U' = L_2$, and hence, $U = L_1 \oplus L_2$. But, this is clearly not Φ-invariant, in contradiction with our assumption. $\square$

We now arrive at the following description of the critical submanifolds.

Proposition 3.12. *The critical submanifolds of f corresponding to critical points of type $(1,1,1)$ are indexed by pairs of integers (m_1, m_2) satisfying $m_1 \geqslant 0$, $m_2 \geqslant 0$, $m_1 + 2m_2 < 6g - 6$, and $2m_1 + m_2 < 6g - 6$. For each pair (m_1, m_2), the critical submanifold $N_{m_1 m_2}$ is given by the pull-back diagram*

$$\begin{tikzcd} N_{m_1 m_2} \arrow[r, "h"] \arrow[d, "\pi"'] & \mathrm{Jac}^{l_2}(\Sigma) \arrow[d, "\pi'"'] \\ S^{m_1}(\Sigma) \times S^{m_2}(\Sigma) \arrow[r, "g"] & \mathrm{Jac}^{r}(\Sigma) \end{tikzcd}$$

where $r = m_1 - m_2 + d$. The map π is defined by taking a Higgs bundle of the form (3.4) to the pair of effective divisors $((\phi_1), (\phi_2))$, while the map h takes the Higgs bundle to L_2. The map g is defined by $g(D_1, D_2) = [D_1 - D_2] \otimes \Lambda_0$ for effective divisors D_i of degree m_i and finally, the map π' is the 3^{2g}-fold covering of the Jacobian given by raising a linebundle to its third power.

Proof. The line bundles M_1 and M_2 do not quite determine L_1, L_2, and L_3. However, a small calculation shows that

$$L_1 = M_1^{-1} L_2 K, \tag{3.12}$$
$$L_3 = M_2 L_2 K^{-1}, \tag{3.13}$$
$$L_2^3 = M_1 M_2^{-1} \Lambda_0. \tag{3.14}$$

Furthermore, the group of automorphisms of E is $\mathbb{C}^\times \times \mathbb{C}^\times$, acting via

$$(\lambda_1, \lambda_2) \mapsto \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1^{-1}\lambda_2^{-1} \end{pmatrix}$$

so, the action on the Higgs field is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ \lambda_1^{-1}\lambda_2\phi_1 0 & 0 & 0 \\ 0 & \lambda_1^{-1}\lambda_2^{-2}\phi_2 & 0 \end{pmatrix}.$$

Therefore, the isomorphism class of the Higgs bundle is determined by the projective classes of ϕ_1 and ϕ_2.

Combining these pieces of information, we see that

$$\pi\colon N_{m_1 m_2} \to S^{m_1}\Sigma \times S^{m_2}\Sigma$$

is the 3^{2g}-fold covering given by pulling back the covering

$$\begin{aligned} \mathrm{Jac}^{l_2}(\Sigma) &\to \mathrm{Jac}^{r}(\Sigma) \\ L_2 &\mapsto L_2^3. \end{aligned}$$

□

Finally, we need to calculate the Betti numbers of $N(m_1, m_2)$.

Proposition 3.13. *The Poincaré polynomial of the component $N(m_1, m_2)$ of the critical submanifold $N(4(g-1)-m)$ is*

$$P_t(N(m_1, m_2)) = P_t(S^{m_1}\Sigma) P_t(S^{m_2}\Sigma) + \binom{2g-2}{m_1}\binom{2g-2}{m_2}(3^{2g}-1)t^m.$$

Proof. $H^*(N(m_1, m_2); \mathbb{R})$ is the $S_{m_1} \times S_{m_2}$-invariant part of $H^*(\tilde{\Sigma}^m; \mathbb{R})$, where $\tilde{\Sigma}^m$ is the covering

$$(\mathbb{Z}/3)^{2g} \to \tilde{\Sigma}^m \to \Sigma^m$$

of Σ^m induced by the composite $\Sigma^m \to S^{m_1}\Sigma \times S^{m_2}\Sigma \xrightarrow{f} \mathrm{Jac}(\Sigma)$. The fundamental group of Σ^m is

$$\pi_1\Sigma^m = \pi_1\Sigma \oplus \cdots \oplus \pi_1\Sigma,$$

the direct sum of m copies of $\pi_1\Sigma$, and we denote the $2g$ generators of the rth copy by $\{\alpha_i^r\}_{i=1}^{2g}$. In the case $m = 1$, the map $f\colon \Sigma \to \mathrm{Jac}(\Sigma)$ induces an isomorphism

$$f^*\colon H^1(\mathrm{Jac}(\Sigma);\mathbb{Z}) \xrightarrow{\cong} H^1(\Sigma;\mathbb{Z}),$$

from which it follows that $\alpha_i^r \in \pi_1\Sigma^m$ acts on the covering

$$(\mathbb{Z}/3)^{2g} \to \tilde{\Sigma}^m \to \Sigma^m$$

by permuting the elements of the ith copy of $\mathbb{Z}/3$ cyclically. Consequently $H^*(\tilde{\Sigma}^m;\mathbb{R}) = H^*(\Sigma^m;\mathcal{F})$, where $\mathcal{F}$ is a local coefficient defined as follows. Let W_i' be the $\pi_1\Sigma^m$-representation in $\mathbb{R}^3$ defined by letting α_i^r permute the basis vectors of $\mathbb{R}^3$, and letting the other α_i^r act trivially. Then, the local coefficient system $\mathcal{F}$ is given by the representation $W_1' \otimes \cdots \otimes W_{2g}'$. Note that W_i' splits as $W_i' = \mathbb{R} \oplus W_i$, where α_i^r acts trivially on $\mathbb{R}$ and rotates the 2-dimensional space W_i through an angle of $\frac{2\pi}{3}$. Thus, $\mathcal{F}$ is of the form

$$\mathcal{F} = \bigoplus_{j=0}^{2g} \mathcal{F}_j,$$

where $\mathcal{F}_0$ is given by the trivial 1-dimensional real representation and $\mathcal{F}_j$ is given by the representation

$$\bigoplus_{i_1<\cdots<i_j} W_{i_1} \otimes \cdots \otimes W_{i_j}.$$

In the following, we use the notation $W_I = W_{i_1} \otimes \cdots \otimes W_{i_j}$ for a multi-index $I = (i_1, \ldots, i_j)$, with $|I| = j$, and we denote the local coefficient system corresponding to the representation W_I by $\mathcal{W}_I$.

The group $H^*(\Sigma;\mathcal{F}_0)$ is just the ordinary real cohomology of Σ. To calculate $H^*(\Sigma^m;\mathcal{F}_j)$ for $j \geqslant 1$, we use the fact that

$$H^0(\Sigma;\mathcal{W}_I) = \{w \in W_I : \xi w = w \quad \text{for all } \xi \in \pi_1\Sigma\},$$

the set of points in W_I fixed by $\pi_1\Sigma$ (see [32, pp. 275–276]). From this, we see that $H^0(\Sigma;\mathcal{W}_I) = 0$ and hence, by Poincaré duality, that $H^2(\Sigma;\mathcal{W}_I) = 0$. By induction on m, it follows that the cohomology of Σ^m with local coefficients $\mathcal{W}_I$ is concentrated in dimension m. The same is certainly true of the cohomology of $S^{m_1}\Sigma \times S^{m_2}\Sigma$, because the cohomology of this space is just the $S_{m_1} \times S_{m_2}$-invariant part of the cohomology of $\tilde{\Sigma}^m$ and, therefore,

$$\dim(H^m(S^{m_1}\Sigma \times S^{m_2}\Sigma); \mathcal{W}_I) = |\chi(S^{m_1}\Sigma)||\chi(S^{m_2}\Sigma)| \dim(W_I).$$

The Euler characteristic of $S^{m_i}\Sigma$ can be calculated to be

$$\chi(S^{m_i}\Sigma) = (-1)^{m_i}\binom{2g-2}{m_i},$$

by Macdonald's formula [22, p.322] for the Poincaré polynomial of the symmetric product of a Riemann surface. This finishes the proof. $\square$

3.5 Calculation of the Morse Indices

Proposition 3.14. *The critical submanifolds for the function f have the following indices.*

i) The critical submanifolds of type $(1,2)$ (described in Proposition 3.7) have Morse index $\lambda_{N_l} = 2(3l - d + 2g - 2)$.

ii) The critical submanifolds of type $(2,1)$ (described in Proposition 3.9) have Morse index $\lambda_{N_l} = 2(d - 3l + 2g - 2)$.

iii) The critical submanifolds of type $(1,1,1)$ (described in Proposition 3.12) have Morse index $\lambda_{N_{m_1 m_2}} = 8g - 8 - 2(m_1 + m_2)$

Proof. We shall only prove i), the other cases being very similar.

Let (E, Φ) be a Higgs bundle of the form (3.2), representing a critical point of type $(1,2)$ and let $\mathbf{T}$ be the tangent space to $\mathcal{M}$ at this point. From the exact sequence (3.1), we have the short exact sequence

$$0 \to \operatorname{coker}(\alpha) \to \mathbf{T} \to \ker(\delta) \to 0.$$

There is an isomorphism

$$\operatorname{Hom}(L, V) \oplus \operatorname{Hom}(V, L) \oplus \operatorname{End}(V) \xrightarrow{\cong} \operatorname{End}_0(E),$$

given by

$$(\eta, \zeta, \xi) \mapsto \begin{pmatrix} -\operatorname{tr}(\xi) & \zeta \\ \eta & \xi \end{pmatrix};$$

and the map of bundles inducing the maps α and δ of (3.1) is

$$\begin{pmatrix} -\operatorname{tr}(\xi) & \zeta \\ \eta & \xi \end{pmatrix} \mapsto \begin{pmatrix} -\zeta\phi & 0 \\ -\operatorname{tr}(\xi)\phi - \xi\phi & \phi\zeta \end{pmatrix}. \tag{3.15}$$

As the infinitesimal gauge transformation ψ giving the decomposition (2.17) has $\operatorname{tr}(\psi) = 0$, we see that ψ has eigenvalues $-i\frac{2}{3}$ on L and $i\frac{1}{3}$ on V and from Section 2.3.2, we know that an eigenvalue in of ψ on $\operatorname{coker}(\alpha)$ corresponds to the eigenvalue $1 - n$ of H_f and, that an eigenvalue im on $\ker(\delta)$ corresponds to $-m$. Thus, we see that H_f can have no negative eigenvalues on $\operatorname{coker}(\alpha)$ while the subspace of $H^1(\operatorname{End}_0(E))$, which can give H_f a negative eigenvalue is $H^1(\operatorname{Hom}(L, V))$. But, from (3.15), we see that δ restricted to this subspace is zero and hence, the index of the critical submanifold N_l is

$$\lambda_{N_l} = \dim_{\mathbb{R}} H^1(\Sigma; \operatorname{Hom}(L, V)).$$

Next, we show that $H^0(\Sigma; \mathrm{Hom}(L, V))$ vanishes. A non-zero element of this space would define a non-trivial map $L \to L'' \subset V$ for some line bundle L'', and hence, $-\deg(L) + \deg(L'') \geqslant 0$. On the other hand $\deg(L'') < \frac{1}{3}d$, by stability, and we also know that $\deg(L) > \frac{1}{3}d$. Hence, $-\deg(L) + \deg(L'') < 0$, which is a contradiction. Therefore,, we can apply the Riemann-Roch formula, to get

$$\begin{aligned} \lambda_{N_l} &= 2(2l - v + 2g - 2) \\ &= 2(3l - d + 2g - 2), \end{aligned}$$

which finishes the proof. $\square$

Chapter 4

The Moduli Space of Flat $\mathrm{Sp}(2n,\mathbb{R})$-connections

In this chapter, we study the moduli space of flat reductive $\mathrm{Sp}(2n,\mathbb{R})$-connections, using the Higgs bundle point of view. In Section 4.1 we give an easy proof of a well-known Milnor-Wood type inequality. The main results of the chapter are in Section 4.2; here we study the connected components of the moduli space of flat reductive $\mathrm{Sp}(4,\mathbb{R})$-connections. In the final Section 4.3 we discuss a possible application of our results to 4-manifolds, which are fibred over a surface.

4.1 Milnor-Wood Type Inequalities

4.1.1 A Proof Using Higgs Bundles

Let P be a $\mathrm{Sp}(2n,\mathbb{R})$-bundle on Σ. This has a characteristic number d, which is the first Chern class of the bundle obtained by a reduction of the structure group to the maximal compact subgroup $\mathrm{U}(n)$. Milnor-Wood type inequalities give bounds on $|d|$, when P is flat. The inequality proved by Milnor [23] concerns $\mathrm{SL}(2,\mathbb{R})$-bundles, while Wood [33] considered $\mathrm{SU}(1,1)$-bundles. Dupont [14] found a bound for any semi-simple group with finite center. However, the inequality of Theorem 4.1 below for $G = \mathrm{Sp}(2n,\mathbb{R})$ is sharper than his. Using the ideas of Gromov [18], Domic and Toledo [12] proved a general result for mappings of a surface into manifolds covered by bounded symmetric domains, and also proved that their inequality is best possible. In particular, their work implies Theorem 4.1 below. Hitchin obtained a proof in the case of flat reductive $\mathrm{SL}(2,\mathbb{R})$-bundles, using Higgs bundles, in [19]. Similarly, we obtain an inequality for flat reductive $\mathrm{Sp}(2n,\mathbb{R})$-bundles (and the same argument works in the case of $\mathrm{SU}(p,p)$-bundles as well).

Theorem 4.1 (Domic-Toledo, Dupont, Milnor, Wood). *Suppose P is a flat reductive $\mathrm{Sp}(2n,\mathbb{R})$-bundle. Then the characteristic number d satisfies the inequality*

$$|d| \leqslant n(g-1).$$

Proof. From Section 2.2.2 we know that flat reductive $\mathrm{Sp}(2n,\mathbb{R})$-bundles correspond to poly-stable Higgs bundles of the form (2.9). The characteristic number is $d = \langle c_1(V), [\Sigma]\rangle$.

Without loss of generality we can assume that $d > 0$. In this case $c \neq 0$, as otherwise V would be Φ-invariant, and therefore violate the stability condition. Let U be the subbundle of V^*, such that $U \otimes K$ is generically the image of c. Similarly, let $U' \subset V$ be the subbundle, which is generically the kernel of c. Then the bundles U' and $V \oplus U$ are both Φ-invariant. We therefore get the following inequalities from semi-stability:

$$\deg(U') \leqslant 0 \tag{4.1}$$
$$d + \deg(U) \leqslant 0. \tag{4.2}$$

Note that these inequalities also hold in the case when $U' = 0$ and $U = V^*$. Next, we note that c induces a non-trivial global section of the linebundle

$$\det(V/U')^{-1} \otimes \det(U \otimes K),$$

which therefore has positive degree, i.e.

$$\deg(U') - d + \deg(U) + (2g-2)\,\mathrm{rk}(c) \geqslant 0. \tag{4.3}$$

Combining this with the inequalities (4.1) and (4.2), we obtain

$$d \leqslant (g-1)\,\mathrm{rk}(c), \tag{4.4}$$

so $d \leqslant n(g-1)$ as claimed. □

Remark 4.2. The above proof gives some additional information. From (4.4) it follows that $\mathrm{rk}(c) = n$ for $d > (n-1)(g-1)$. In particular, in the extremal case $d = n(g-1)$, we have $\mathrm{rk}(c) = n$, and furthermore equality holds in (4.3). Hence, $\det(c)$ is a non-zero section of a linebundle of degree 0, and we conclude that $c\colon V \to V^* \otimes K$ is an isomorphism.

4.1.2 The Extremal Case

The Higgs bundles of the form (2.9) are examples of Q-bundles, where the the quiver Q has two vertices and two arrows, one connecting the vertices in each direction. Thus, it is of the special form considered in Section 2.4 and we can therefore apply Theorem 2.4. This is particularly useful in the extremal case $d = n(g-1)$, where we obtain an identification of the subspace $\mathcal{M}_{n(g-1)} \subset \mathcal{M}_{\mathrm{Sp}(2n,\mathbb{R})}$ of Higgs bundles with $d = n(g-1)$, with a moduli space of rank n Higgs bundles.

Let (E, Φ) be a Higgs bundle of the form (2.9) with $d = n(g-1)$. Choosing a square root L_0 of the canonical bundle on Σ, we can define a rank n vector bundle W by

$$W = V \otimes L_0^{-1},$$

and we can define $C \in H^0(\Sigma; S^2W^*)$ and $\phi \in H^0(\Sigma; \mathrm{End}(W) \otimes K^2)$ by

$$C = c \otimes 1_{L_0},$$

and

$$\phi = (b \otimes 1_{L_0^{-1}}) \circ (c \otimes 1_{L_0}).$$

Note that ϕ is symmetric with respect to the quadratic form C.

From Remark 4.2 we know that c is an isomorphism when (E, Φ) is poly-stable, and thus we can recover (E, Φ) from this data. Therefore the set of isomorphism classes of Higgs bundles of the form (2.9) is equal to the set of isomorphism classes of Higgs bundles

$$(W, C, \phi), \tag{4.5}$$

where W has a non-degenerate quadratic form C, and the Higgs field Φ is twisted by K^2 and symmetric with respect to C. There is an obvious stability condition for (W, C, ϕ), namely that

$$\mu(U) < \mu(W) \tag{4.6}$$

for all ϕ-invariant subbundles U of W. Next, we shall prove that (W, C, ϕ) is stable, if and only if (E, Φ) is.

Theorem 4.3. *The subspace $\mathcal{M}_{n(g-1)} \subset \mathcal{M}_{\mathrm{Sp}(2n,\mathbb{R})}$ of Higgs bundles of the form* (2.9)*, with $d = n(g-1)$ is isomorphic to the moduli space of poly-stable Higgs bundles of the form* (4.5)*.*

Proof. We have to prove that (E, Φ) is stable if and only if (W, C, ϕ) is. From Theorem 2.4 we know that stability of (E, Φ) is equivalent to stability of the Q-bundle $\mathbf{E} = (\underline{E}, \underline{\Phi})$. Thus, all we need to prove is that $\mathbf{E}$ is stable if and only if (W, C, ϕ) is. Because stability is unaffected by tensoring with a line bundle, we can equally well prove that $(V, b \circ c)$ is stable. Note, that $\mu(V) = g - 1$.

Assume $\mathbf{E}$ is a stable Q-bundle. Let $U \subset V$ be a ϕ-invariant subbundle. Let $U' \subset V^*$ be the subbundle such that $U' \otimes K$ is generically the image of U under c. Then b maps U' to U, because of the ϕ-invariance of U. Hence, $\mathbf{F} = (\{U, U'\}, \{b, c\})$ defines a Q-subbundle of $\mathbf{E}$, and it follows that

$$\mu(\mathbf{F}) < \mu(\mathbf{E}). \tag{4.7}$$

But, as c is an isomorphism

$$\begin{aligned}\mu(\mathbf{E}) &= \mu(E)\\ &= \mu(V \oplus V \otimes K^{-1})\\ &= \mu(V) - (g-1),\end{aligned}$$

and similarly $\mu(\mathbf{F}) = \mu(U) - (g-1)$. Therefore $\mu(U) < \mu(V)$ and so, (W, C, ϕ) is stable.

Conversely, assume that (W, C, ϕ) is stable. Let $\mathbf{F} = (\{U, U'\}, \{b, c\})$ be a Q-subbundle of $\mathbf{E}$. Let $\tilde{U} \subset V^*$ be the subbundle which is generically the image of $U' \otimes K$ under c^{-1}. Both U and $\tilde{U}$ are ϕ-invariant subbundles of V, because $\mathbf{F}$ is a Q-subbundle. Hence, $\mu(U) < \mu(V)$ and $\mu(U') < \mu(V)$, by stability of (W, C, ϕ). Recalling that $\mu(V) = g-1$ and $\mu(\tilde{U}) = \mu(U') - (2g-2)$, we get

$$\mu(U) < g - 1, \tag{4.8}$$

and

$$\mu(U') < -(g-1). \tag{4.9}$$

Note also that

$$\mathrm{rk}(U') \geqslant \mathrm{rk}(U), \tag{4.10}$$

because c is an isomorphism, and the image of U under c is contained in $U' \otimes K$, by the assumption that $\mathbf{F}$ is a Q-subbundle. Combining (4.8), (4.9), and (4.10), we get:

$$\begin{aligned}\mu(\mathbf{F}) &= \mu(U \oplus U')\\ &= \frac{\mathrm{rk}(U)}{\mathrm{rk}(U \oplus U')}\mu(U) + \frac{\mathrm{rk}(U')}{\mathrm{rk}(U \oplus U')}\mu(U')\\ &< \frac{\mathrm{rk}(U) - \mathrm{rk}(U')}{\mathrm{rk}(U \oplus U')}(g-1)\\ &\leqslant 0.\end{aligned}$$

Of course, $\mu(\mathbf{E}) = 0$ and hence, the proof finished. $\square$

4.2 The Components of $\mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$

In this section, we shall examine the number of connected components of the space $\mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$ of flat reductive $\mathrm{Sp}(4,\mathbb{R})$-connections by means of its interpretation as the moduli space of poly-stable Higgs bundles of the form (2.9).

In general, the reduction to $\mathrm{U}(n) \subset \mathrm{Sp}(2n,\mathbb{R})$ already provides some information. The first Chern class ($d = \langle c_1(V), [\Sigma]\rangle$) is a continuous integer-valued function on $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{R})}$ and thus separates it into subspaces which are unions of different connected components. From Theorem 4.1, we know that $|d| \leqslant n(g-1)$ and we denote the subspace corresponding to d by $\mathcal{M}_d$, for $d = -n(g-1), \ldots, n(g-1)$. Note that $\mathcal{M}_{-d} \cong \mathcal{M}_d$ by taking the vector bundle V to its dual.

We specialise to the case of $\mathrm{rk}(V) = 2$. We shall find the connected components of the subspace $\mathcal{M}_{2g-2}$. We conjecture that the spaces $\mathcal{M}_d$ are connected for $|d| < 2g-2$, and we prove this in the case $d = 0$. We discuss two possible approaches to a general proof in Subsection 4.2.7.

4.2.1 Statement of the Result

In the extremal case $d = 2g-2$, Theorem 4.3 gives an identification of $\mathcal{M}_{2g-2}$ with the moduli space of poly-stable Higgs bundles of the form (4.5), with W a bundle of rank 2. Recall that V and W are related by

$$V = W \otimes L_0,$$

where L_0 is a square root of the canonical bundle.

The existence of the quadratic form C on W means that the structure group is $\mathrm{O}(2,\mathbb{C})$. The maximal compact subgroup of $\mathrm{O}(2,\mathbb{C})$ is $\mathrm{O}(2)$ and, therefore, we have the Stiefel-Whitney classes w_1 and w_2 as topological invariants.

The first Stiefel-Whitney class can be seen in holomorphic terms as follows: the quadratic form C gives an isomorphism $(\Lambda^2 W)^2 \cong \mathcal{O}$; hence, $\Lambda^2 W$ gives an element of $H^1(\Sigma;\mathbb{Z}/2)$, and it is easy to see that this element is $w_1(W)$. It follows that $\Lambda^2 W = \mathcal{O}$ if and only if $w_1(W) = 0$. This, in turn, is equivalent to the existence of a reduction of structure group to $\mathrm{SO}(2,\mathbb{C}) \subset \mathrm{O}(2,\mathbb{C})$. Using the identification $\mathbb{C}^\times \cong \mathrm{SO}(2,\mathbb{C})$ via

$$\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

we see that this happens exactly when W decomposes as a direct sum

$$W = L \oplus L^{-1},$$

and C is of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to this decomposition. Now it is clear that, in this case, $w_2(W)$ is given by

$$w_2 = c_1(L) \mod 2.$$

By interchanging L with its dual if necessary, we may assume that $\deg(L) \geqslant 0$. Furthermore, when $\deg(L) > 0$, the Higgs field ϕ must induce a non-zero holomorphic map

$$L \to L^{-1}K,$$

because otherwise $L \subset W$ would violate stability. Hence, we have

$$\deg(L) \leqslant 2g - 2.$$

We, therefore, have a decomposition of $\mathcal{M}_{2g-2}$ into subspaces, each of which is a union of connected components, as follows:

$$\mathcal{M}_{2g-2} = \Big(\bigcup_{u,v} \mathbf{M}_u^v\Big) \cup \Big(\bigcup_{l=0}^{2g-2} \mathbf{M}_0^l\Big),$$

where $\mathbf{M}_u^v$ is the moduli space of poly-stable Higgs bundles (W, C, ϕ) with $w_1(W) = u \in H^1(\Sigma; \mathbb{Z}/2) - \{0\}$ and $w_2(W) = v \in H^2(\Sigma; \mathbb{Z}/2)$, and where $\mathbf{M}_0^l$ is the moduli space of poly-stable Higgs bundles (W, C, ϕ) with $w_1(W) = 0$ and $\deg(L) = l$.

We can now state our main result.

Theorem 4.4. *The space $\mathcal{M}_0$ is connected. For $d = 2g - 2$, the spaces $\mathbf{M}_a^b$ and $\mathbf{M}_0^l$ are connected.*

Remark 4.5. As we shall see later (Remark 4.25), the subspaces $\mathcal{M}_d$ are non-empty. Therefore, Theorem 4.4 shows that $\mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$ has at least $2^{2g+2} + 8g - 11$ connected components.

Remark 4.6. For general n, the reduction of structure group to $\mathrm{O}(n, \mathbb{C})$ is, of course, still valid, and gives the Stifel-Whitney classes w_1 and w_2 as further topological invariants. Knowing that the corresponding subspaces were non-empty, would give us a lower bound on the number of components of $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{R})}$.

4.2.2 Strategy of Proof

We shall again use the function $f = \frac{1}{2}\|\Phi\|^2$ to analyse the connectivity properties of $\mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$. This space is not smooth, so it is not clear how to do Morse theory on it. However, the function f is proper so, in order to show that a subspace of $\mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$ is connected, it is enough to show that the subspace of local minima of f on this space is connected.

Therefore, our strategy is first, to find the local minima of f on each of the subspaces $\mathbf{M}_u^v$ and $\mathbf{M}_0^l$, and then, to show that in each of these spaces, the subspace of local minima is connected.

In Subsection 4.2.3, we find the critical points of f and determine which of them are local minima. In order to show that the subspaces of minima are connected, we need the theory of the spectral curve and we give a review of this, following Beauville, Narasimhan, and Ramanan [4], in Subsection 4.2.4. Finally, we prove that the spaces of minima are connected in Subsections 4.2.5 and 4.2.6.

4.2.3 The Structure of the Critical Points

Clearly, any (E,Φ) with $\Phi = 0$ is fixed under the circle action and corresponds to the absolute minimum of f. When (E,Φ) is of the form (2.9), it follows from stability that $\deg(V) = 0$.

Let (E,Φ) be a stable Higgs bundle of the form (2.9) with $\Phi \neq 0$, which is a critical point of the circle action; it is thus of the form (2.17). The infinitesimal gauge transformation giving this decomposition is of the form

$$\Psi = \begin{pmatrix} \psi & 0 \\ 0 & -\psi^T \end{pmatrix}$$

for a non-zero $\psi \in H^0(\Sigma; \mathrm{End}(V))$ and hence, the decomposition (2.17) respects the decomposition $E = V \oplus V^*$. There are now two possibilities: either ψ is multiple of the identity or, it has two distinct eigenvalues.

When the eigenvalues of ψ are distinct, we can split V in eigenbundles for ψ. Thus, $V = L_1 \oplus L_2$, where L_i is an eigenbundle for ψ corresponding to the eigenvalue im_i. In view of the form of Φ and the fact that consecutive eigenvalues of Ψ differ by i, we see that $m_1 = -3/2$, $m_2 = 1/2$ and that the components b and c of Φ take the following form:

$$b = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 & \tilde{c} \\ \tilde{c} & 0 \end{pmatrix},$$

where $\tilde{b} \in H^0(\Sigma; L_2^2 K)$ and $\tilde{c} \in H^0(\Sigma; L_1^{-1} L_2^{-1} K)$. We define Higgs bundles of type $(1,1,1,1)$ to be Higgs bundles of this form. Note that stability of (E,Φ)

implies that $\tilde{b}$ and $\tilde{c}$ are non-zero, as otherwise (E, Φ) would decompose as a direct sum of lower rank bundles.

When $\psi = im$ is a multiple of the identity, again using the fact that consecutive eigenvalues of Ψ differ by i, we see that $m = -1/2$. Now Φ is of the form

$$\Phi = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

We define a Higgs bundle to be of type $(2, 2)$, if it is of this kind.

When (E, Φ) is only poly-stable, it decomposes as a direct sum of lower rank Higgs bundles of the same form. In other words, $E = E_1 \oplus E_2$ with $\Phi = \Phi_1 + \Phi_2$, where $\Phi_\nu \colon E_\nu \to E_\nu K$ for $\nu = 1, 2$. Furthermore,

$$E_\nu = L_\nu \oplus L_\nu^{-1}$$

for a linebundle L_ν, and Φ_ν is of the form

$$\Phi_\nu = \begin{pmatrix} 0 & b_\nu \\ c_\nu & 0 \end{pmatrix}$$

with respect to this decomposition. Without loss of generality, we may assume that $\deg(L_\nu) \geqslant 0$. If (E, Φ) is a critical point, it follows that each of (E_ν, Φ_ν) are of the form (2.17). Therefore, Φ_ν is of the form

$$\Phi_\nu = \begin{pmatrix} 0 & 0 \\ c_\nu & 0 \end{pmatrix},$$

and we see that these critical points may be considered as critical points of type $(2, 2)$ as well. Note, in particular, that critical points of type $(1, 1, 1, 1)$ are smooth points of the moduli space.

Critical Points of Type $(1, 1, 1, 1)$

We consider a critical point (E, Φ) of type $(1, 1, 1, 1)$. Letting $\psi = \left(\begin{smallmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{smallmatrix}\right)$, the sheaf maps α_1 and α_2 of (2.11) and (2.12) take the form

$$\alpha_1(\psi) = \begin{pmatrix} 0 & \tilde{b}\psi_{12} \\ \tilde{b}\psi_{12} & 2\tilde{b}\psi_{22} \end{pmatrix} \tag{4.11}$$

and

$$\alpha_2(\psi) = \begin{pmatrix} -2\tilde{c}\psi_{21} & -\tilde{c}(\psi_{11} + \psi_{22}) \\ -\tilde{c}(\psi_{11} + \psi_{22}) & -2\tilde{c}\psi_{12} \end{pmatrix}. \tag{4.12}$$

In the following proposition, we collect some simple numerical bounds on these Higgs bundles.

Proposition 4.7. *Let (E, Φ) be a stable Higgs bundle of the above type and put $d_1 = \deg(L_1)$ and $d_2 = \deg(L_2)$. Then the following inequalities are satisfied.*

$$\begin{aligned} \tfrac{1}{2}d &< d_1 \leqslant d + g - 1 \\ 1 - g &\leqslant d_2 < d_1. \end{aligned}$$

Proof. First, observe that the linebundles $L_1^{-1}L_2^{-1}K$ and L_2^2K have non-zero holomorphic sections (viz. $\tilde{c}$ and $\tilde{b}$). Hence, $d_1 + d_2 \leqslant 2g - 2$ and $d_2 \leqslant 1 - g$. The first of these inequalities is, of course, just the Milnor-Wood type inequality of Theorem 4.1. Furthermore, the bundles L_1^{-1} and $L_2 \oplus L_1^{-1}$ are Φ-invariant subbundles of E and hence, we get, by stability, $d_1 > 0$ and $d_1 > d_2$. Combining these inequalities gives the stated result. □

We can now determine under which circumstances these critical points can be minima of f.

Proposition 4.8. *A stable Higgs bundle of type $(1, 1, 1, 1)$ corresponds to a local minimum of f if and only if $d = 2g - 2$.*

Proof. From the exact sequence (2.10), we have the short exact sequence

$$0 \to \operatorname{coker}(\alpha) \to \mathbf{T} \to \ker(\delta) \to 0.$$

From Subsection 2.3.2, we know that an eigenvalue in of ψ on $\operatorname{coker}(\alpha)$ corresponds to the eigenvalue $1 - n$ of H_f and that an eigenvalue im on $\ker(\delta)$ corresponds to $-m$. Thus, we see that the subspace of

$$H^0(\Sigma; S^2V \otimes K \oplus S^2V^* \otimes K),$$

which can give H_f a negative eigenvalue, is

$$H^0(\Sigma; L_1^{-2}K) \subset H^0(\Sigma; S^2V^* \otimes K),$$

giving the eigenvalue -2. From (4.12), it follows that the corresponding subspace of $\operatorname{coker}(\alpha)$ is the cokernel of the map

$$\eta_*\colon H^0(\Sigma; L_1^{-1}L_2) \to H^0(\Sigma; L_1^{-2}K),$$

induced by the sheaf map

$$\begin{aligned} \eta\colon \mathcal{O}(L_1^{-1}L_2) &\to \mathcal{O}(L_1^{-2}K) \\ s &\mapsto 2\tilde{c}s. \end{aligned}$$

Similarly, we see that the subspace of $\ker(\delta)$, which gives H_f a negative eigenvalue, is the kernel of the map

$$\eta_*\colon H^1(\Sigma; L_1^{-1}L_2) \to H^1(\Sigma; L_1^{-2}K),$$

again giving the eigenvalue -2.

We can use the Riemann-Roch formula to calculate the Morse index,

$$\begin{aligned}\lambda &= h^0(L_1^{-2}K) - h^0(L_1^{-1}L_2) + h^1(L_1^{-1}L_2) - h^1(L_1^{-2}K)\\ &= 2g - 2 - d.\end{aligned}$$

It follows that the Morse index of the critical point is zero if and only if $d = 2g - 2$. □

Critical Points of Type $(2, 2)$

Here, we consider critical points corresponding to Higgs bundles of the form

$$E = V \oplus V^*, \tag{4.13}$$

with

$$\Phi = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

Using Hitchin's equations, the value of f at a critical point of this kind can be calculated to be

$$\begin{aligned}f(E, \Phi) &= \tfrac{1}{2}\|c\|^2\\ &= 2\pi c_1(V).\end{aligned}$$

Remark 4.9. It follows from this that, when $c_1(V) = 0$, we must have $c = 0$. Together with Proposition 4.8 this shows that the the subvariety of local minima, $N_0 \subset \mathcal{M}_0$, is the moduli space of poly-stable Higgs bundles (E, Φ) of the form (2.9) with $\deg(V) = 0$ and $\Phi = 0$.

For $0 < d \leqslant 2g - 2$, we denote the corresponding critical set by N_d. Thus, the subvariety $N_d \subset \mathcal{M}_d$ is the moduli space of poly-stable Higgs bundles of the form (4.13) with $\deg(V) = d$. The action of ψ on $S^2V \otimes K$ is multiplication by $-i$, the action on $S^2V^* \otimes K$ is multiplication by i, and the action on $\mathrm{End}(V)$ is multiplication by 0. From Subsection 2.3.2, we know that the corresponding eigenvalues of H_f are 2, 0, and 0, respectively. Thus, the Morse index of N_d is necessarily 0 and hence, it is a local minimum of f. From Proposition 4.8 it follows that N_d is the whole subvariety of local minima of f on $\mathcal{M}_d$, for $0 < d < 2g - 2$

Next, we shall calculate the dimension of N_d. At a smooth point, this is the dimension of the subspace of the tangent space $\mathbf{T}$ on which H_f has eigenvalue 0. Consider the exact sequence (2.10). Note that, in this case, the maps α and δ are induced by the sheaf map

$$\begin{aligned} \eta\colon \mathcal{O}(\mathrm{End}(V)) &\to \mathcal{O}(S^2V^* \otimes K) \\ \psi &\mapsto -\psi^T c - c\psi. \end{aligned}$$

At a smooth point, α is injective and δ is surjective. Note, in particular, that $H^1(\Sigma; S^2V \otimes K)$ vanishes. Thus, we can calculate the dimension from the Riemann-Roch Formula as follows:

$$\begin{aligned} \dim(N_d) &= h^0(S^2V^* \otimes K) - h^0(\mathrm{End}(V)) + h^1(\mathrm{End}(V)) - h^1(S^2V^* \otimes K) \\ &= 7g - 7 - 3d. \end{aligned}$$

Note that (E, Φ) can be recovered from (V, c). Using Theorem 2.4, we can formulate the stability condition in terms of this pair.

Proposition 4.10. *The Higgs bundle (E, Φ) is stable if and only if*

$$\deg(F) + \deg(G) < d,$$

for all proper subbundles F and G of V, which are orthogonal with respect to c (here we think of c as a K-valued quadratic form on V).

Proof. It follows from Theorem 2.4 that (E, Φ) is stable if and only if

$$\deg(F) + \deg(\tilde{G}) < 0, \tag{4.14}$$

for all pairs (F, G), such that $F \subset V$, $\tilde{G} \subset V^*$ and $c(F) \subset \tilde{G} \otimes K$. Define $G = (V^*/\tilde{G})^* \subset V$; then,

$$\deg(G) = d + \deg(\tilde{G}).$$

Note that F and G are orthogonal with respect to c if and only if $c(F) \subset \tilde{G} \otimes K$. When (E, Φ) is stable, it follows from (4.14) that

$$\deg(F) + \deg(G) < d.$$

Clearly, we can go the other way and hence, the proposition is proved. □

Remark 4.11. Note that (V, c) is poly-stable, but not stable, when it decomposes as a direct sum of stable pairs of lower rank. In particular, there is no condition on the degrees of these bundles. This notion is, of course, equivalent to poly-stability of (E, Φ).

Remark 4.12. There is no need to restrict to the $\mathrm{rk}(V) = 2$ case here; the proof works for V of any rank.

Remark 4.13. One can easily check that, as it should be, the stability condition of the proposition is equivalent to the stability condition (4.6) when $d = n(g-1)$ and $\phi = 0$.

4.2.4 The Spectral Curve

Consider a pair (V, c), where V is a rank 2 bundle of degree d and

$$c \in H^0(\Sigma; S^2V^* \otimes K)$$

is a quadratic form, which is generically non-degenerate. Using the isomorphism

$$\begin{aligned} V^* &\to V \otimes \Lambda^2 V^* \\ \omega &\mapsto \omega \wedge -, \end{aligned}$$

and putting $L = \Lambda^2 V^* K$, we can alternatively think of c as an element of

$$H^0(\Sigma; \mathrm{End}_0(V) \otimes L).$$

Here $\mathrm{End}_0(V)$ denotes the trace-free endomorphisms of V. In this way, (V, c) can be considered as an ordinary Higgs bundle.

From either point of view, the determinant of c is a non-zero section

$$\det(c) \in H^0(\Sigma; L^2).$$

We shall consider the spectral curve associated to (V, c). The spectral curve was introduced by Hitchin [20], but here we shall follow the exposition of Beauville, Narasimhan, and Ramanan [4].

Consider the compactification of L:

$$\pi \colon \mathbb{P}(\mathcal{O} \oplus L) \to \Sigma.$$

The hyperplane bundle along the fibres is a linebundle on the total space of $\mathbb{P}(\mathcal{O} \oplus L)$, which we denote by $\mathcal{O}(1)$. This has a canonical section, y, given by the section

$$(1, 0) \in H^0(\Sigma; (\mathcal{O} \oplus L)^*);$$

in other words, by projecting $\mathcal{O} \oplus L \to \mathcal{O}$. Similarly, there is a canonical section, x, of $\pi^*L \otimes \mathcal{O}(1)$, given by the section

$$(0, 1) \in H^0(\Sigma; L \oplus \mathcal{O});$$

in other words, by projecting $\mathcal{O} \oplus L \to L$.

The spectral curve $\tilde{\Sigma}$ is defined as the zero locus of the section

$$s = \pi^* \det(c) \cdot y^2 + x^2$$

of $\pi^* L^2 \otimes \mathcal{O}(2)$. The zeros of x and y are disjoint in $\mathbb{P}(\mathcal{O} \oplus L)$ so, the restriction of y to $\tilde{\Sigma}$ is nowhere vanishing. Thus, $\mathcal{O}(1)$ restricted to $\tilde{\Sigma}$ is trivial and the restriction of x can be considered as a section

$$x \in H^0(\tilde{\Sigma}; \pi^* L).$$

Furthermore, the trace of c vanishes, so the section s can be thought of as the characteristic polynomial of c in homogeneous coordinates x and y.

The double cover $\tilde{\Sigma} \xrightarrow{\pi} \Sigma$ is, in general, ramified, and the ramification divisor is $D = (\det(c))$. Furthermore, when $\det(c)$ has simple zeros, $\tilde{\Sigma}$ is smooth (see [4]). From now on, we shall assume that this is the case. The genus $\tilde{g}$ of $\tilde{\Sigma}$ can be calculated from the Riemann-Hurwitz formula:

$$\begin{aligned} 2 - 2\tilde{g} &= 4 - 4g - 2\deg(L) \\ &= 8 - 8g + 2d, \end{aligned}$$

and thus,

$$\tilde{g} = 4g - 3 - d.$$

Beauville, Narasimhan, and Ramanan show that there is a 1–1 correspondence between isomorphism classes of linebundles $M \to \tilde{\Sigma}$ and isomorphism classes of pairs (V, c). The linebundle M corresponding to (V, c) fits into an exact sequence

$$0 \to M(-D) \to \pi^* V \xrightarrow{\pi^* c - x} \pi^*(V \otimes L) \to M \otimes \pi^* L \to 0.$$

Thus, $M(-D)$ can be thought of as the eigenspace bundle corresponding to the eigenvalue x of c. Clearly, the linebundle $\tau^* M(-D)$ is the eigenspace bundle corresponding to the eigenvalue $-x$. Thinking of $\pi^* c$ as a $\pi^* K_\Sigma$-valued quadratic form on $\pi^* V$, we see that it gives a holomorphic section

$$M(-D) \otimes \tau^* M(-D) \to \pi^* K_\Sigma,$$

which has divisor $2D$. Therefore,

$$M \otimes \tau^* M \cong \pi^* K_\Sigma. \tag{4.15}$$

Conversely, when M satisfies (4.15), the pair (V, c) obtained from M will have c symmetric. We, therefore, have the following version of the result of Hitchin and Beauville, Narasimhan, and Ramanan. We are still working under the assumption that $(\det(c))$ has simple zeros.

Proposition 4.14. *There is a 1–1 correspondence between linebundles M on $\tilde{\Sigma}$ of degree $2g-2$, such that the condition (4.15) is satisfied, and pairs (V, c), where c is a generically non-degenerate K_Σ-valued form on the rank 2 vector bundle V, such that $(\det(c))$ is the ramification divisor of the covering.*

Next, we shall examine to what extent a (V, c), obtained in this way, is stable.

Proposition 4.15. *Let (V, c) be a pair as above, with $0 < d \leqslant 2g-2$ and $\mathrm{rk}(V) = 2$. Suppose that c is generically non-degenerate and that $\det(c)$ has simple zeros. Then, either (V, c) is stable in the sense of Proposition 4.10, or*

$$V \cong L \oplus L^{-1}K$$

for a linebundle L of degree $g-1$, and c is of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to this decomposition. In particular, $d = 2g-2$.

Proof. Because $\mathrm{rk}(V) = 2$ and c is generically non-degenerate, there are only the following two cases to consider.

1. Let L and $L^\perp$ be distinct rank 1 subbundles of V, which are orthogonal with respect to c. Then,

$$\begin{aligned} L \oplus L^\perp &\to V \\ (x, y) &\mapsto x + y \end{aligned}$$

gives a generic isomorphism and hence,

$$\deg(L) + \deg(L^\perp) \leqslant d,$$

with equality only when (V, c) is of the form given above. But this is exactly the stability condition of Proposition 4.10.

2. Let $L \subset V$ be a linebundle which is null with respect to c.

First, suppose that $d = 2g-2$. We can then consider the pair (W, C) instead, and W is an extension

$$0 \to L \to W \to L^{-1} \to 0.$$

In particular, $\Lambda^2 W = \mathcal{O}$ and therefore, $w_1(W) = 0$. Thus, there is a reduction of structure group to $\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^\times$ and, as we have seen earlier, it follows that V is of the form given above.

Thus, we may assume that $d < 2g-2$ and consider the spectral curve $\tilde{\Sigma}$. In the notation of the preceding discussion of the spectral curve, the pull-back π^*L to $\tilde{\Sigma}$ is either equal to $M(-D)$ or $\tau^*M(-D)$. In both cases, it follows that

$$\begin{aligned}\deg(L) &= \tfrac{1}{2}\deg(M(-D)) \\ &= d+1-g \\ &< \tfrac{1}{2}d.\end{aligned}$$

This is again the stability condition of Proposition 4.10 and the proof is finished. ☐

4.2.5 The $d = 0$ case

In this subsection, we shall prove the following theorem.

Theorem 4.16. *The subspace $\mathcal{M}_0 \subset \mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$ is connected.*

Proof. Recall from Remark 4.9, that the subvariety of local minima N_0 is the moduli space of poly-stable Higgs bundles (E,Φ) of the form (2.9) for which $d = 0$ and Φ vanishes. Thus, (E,Φ) is determined by V. It is easy to see that the Higgs bundle (E,Φ) is poly-stable if and only if V is poly-stable in the ordinary sense, and hence, N_0 is the moduli space of poly-stable vector bundles of rank 2 and degree 0. This can also be seen from the point of view of Hitchin's equations: when $\Phi = 0$ the solutions give the moduli space of flat $U(2)$-connections, which by the theorem of Narasimhan and Seshadri is isomorphic to the moduli space of poly-stable vector bundles of rank 2 and degree 0.

This moduli space is well known to be connected (see e.g. Atiyah and Bott [3]) so, from the properness of f, we conclude that $\mathcal{M}_0$ is connected. ☐

4.2.6 The $d = 2g-2$ case

The $w_1(W) = 0$ Case

Here, our aim is to prove the following theorem.

Theorem 4.17. *The subvarieties $\mathbf{M}_0^l \subset \mathcal{M}_{2g-2}$ are connected*

Recall that any (W, C, ϕ) in M_0^l is of the form

$$W = L \oplus L^{-1},$$

with $l = \deg(L)$ and C of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. First, we consider the case of $l > 0$. In this case, the Higgs field ϕ must be non-zero, as otherwise the subbundle $L \subset W$ would violate stability. But any critical point of type $(2,2)$ has $\phi = 0$ so, it follows that all the critical points in $\mathbf{M}_0^l$ for $l > 0$ are of type $(1,1,1,1)$.

From the description of these critical points in Subsection 4.2.3, it follows easily that they correspond to Higgs bundles (W, C, ϕ), which are of the form described above and where, furthermore, ϕ is of the form

$$\phi = \begin{pmatrix} 0 & 0 \\ \tilde{\phi} & 0 \end{pmatrix},$$

with $\tilde{\phi} \in H^0(\Sigma; L^{-2}K^2)$. Using this, it is now easy to give an explicit description of N_0^l.

Proposition 4.18. *The subvariety of local minima $N_0^l \subset \mathbf{M}_0^l$ fits into a pull-back diagram*

$$\begin{array}{ccc} N_0^l & \longrightarrow & \mathrm{Jac}^l(\Sigma) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle L \mapsto L^{-2}K^2} \\ S^{4g-4-2l}\Sigma & \xrightarrow{D \mapsto [D]} & \mathrm{Jac}^{4g-4-2l}(\Sigma), \end{array}$$

where $\pi(W, C, \phi) = (\phi)$.

Proof. The only thing there is to remark is that any (W, C, ϕ), of the form given above, is stable. But, $L^{-1} \subset W$ is the only ϕ-invariant subbundle so, this is obvious. □

From this proposition, it is clear that N_0^l is connected so, from the properness of f, it follows that M_0^l is connected. This proves Theorem 4.17 in the cases $l > 0$.

In the case $l = 0$, we have the following

Proposition 4.19. *Any local minima of f in $\mathbf{M}_0^0$ has $\phi = 0$ and is, therefore, of type $(2,2)$.*

Proof. Suppose we have a critical point of type $(1,1,1,1)$ as above, with $\phi \neq 0$. Then, $L^{-1} \subset W$ is ϕ-invariant and therefore, (W, C, ϕ) is semi-stable, but not stable. Since we are considering the moduli space of poly-stable Higgs bundles, (W, C, ϕ) decomposes as a direct sum of rank 1 Higgs bundles of degree 0. The only subbundles of W of rank 1 and degree 0 are L and L^{-1}, and L is not ϕ-invariant so, we conclude that this situation cannot occur. □

Consequently, we have the following description of the subspace of $\mathbf{M}_0^0$ of local minima of f.

Proposition 4.20. *The subspace $N_0^0 \subset \mathbf{M}_0^0$ of local minima of f is isomorphic to the moduli space of poly-stable (W, C), where W is of the form*

$$W = L \oplus L^{-1},$$

for a linebundle L of degree 0, and C is of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with respect to this decomposition.

Remark 4.21. The pair (W, C) decomposes into a direct sum of linebundles exactly when $L^2 = \mathcal{O}$, and it is then poly-stable, but not stable. However, all other (W, C) are stable.

It follows that there is a surjective continuous map

$$\mathrm{Jac}^0(\Sigma) \to N_0^0,$$

given by taking L to (W, C) of the form given above. Therefore, N_0^0 is connected, finishing the proof of Theorem 4.17.

The $w_1(W) \neq 0$ Case.

In this instance, we prove the following theorem.

Theorem 4.22. *The subvarieties $\mathbf{M}_u^v \subset \mathcal{M}_{2g-2}$ are connected.*

Because any critical point of type $(1,1,1,1)$ has $w_1(W) = 0$, we see that the subvarieties of local minima $N_u^v \subset \mathbf{M}_u^v$ consist of critical points of type $(2,2)$. Recall that for these $b = 0$; in terms of the Higgs bundle (W, C, ϕ), this means that $\phi = 0$. Thus, N_u^v is the moduli space of stable pairs (W, C) with the given characteristic classes. Since $\Lambda^2 W \neq \mathcal{O}$, it follows from Proposition 4.10 that any such pair is stable. Hence, we do not need to worry about stability.

There is a connected double cover $\tilde{\Sigma} \xrightarrow{\pi} \Sigma$ given by

$$w_1(W) \in H^1(\Sigma; \mathbb{Z}/2) = \mathrm{Hom}(\pi_1 \Sigma, \mathbb{Z}/2).$$

Clearly, $\pi^* W = M \oplus M^{-1}$ with $\pi^* C = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Because the cover is unramified, $K_{\tilde{\Sigma}} = \pi^* K_\Sigma$, and $\pi^* L_0$ is a square root of $K_{\tilde{\Sigma}}$. Obviously,

$$\pi^* V = M \pi^* L_0 \oplus (M \pi^* L_0)^{-1} K_{\tilde{\Sigma}}$$

and hence, $\tilde{\Sigma}$ is the spectral curve associated to (V, c). Note that the bundle which was called M in Subsection 4.2.4 is $M\pi^* L_0$ here. Therefore, condition (4.15) becomes

$$\tau^* M = M^{-1}.$$

In other words, $N_u^0 \cup N_u^1$ is the kernel of the map

$$1 + \tau^* \colon \operatorname{Jac}(\tilde{\Sigma}) \to \operatorname{Jac}(\tilde{\Sigma}),$$

where $\tilde{\Sigma}$ is the unramified double cover of Σ given by $u \in H^1(\Sigma; \mathbb{Z}/2)$.

It remains to distinguish between w_2 being equal to 0 or 1. When the cover is unramified, the kernel of $1 + \tau^*$ splits into two components,

$$\ker(1 + \tau^*) = P^+ \cup P^-,$$

each of them a translate of the Prym variety of the covering. It is a classical theorem of Wirtinger, that the function $\delta \colon P^+ \cup P^- \to \mathbb{Z}/2$, defined by

$$\begin{aligned} \delta(M) &= \dim_{\mathbb{C}} H^0(\tilde{\Sigma}; M \otimes \pi^* L_0) \mod 2 \\ &= \dim_{\mathbb{C}} H^0(\tilde{\Sigma}; \pi_* M \otimes L_0) \mod 2, \end{aligned}$$

is constant on each of P^+ and P^- and takes different values on them. For proofs of these facts, see Mumford [24] or [25].

Now, let $F \to \Sigma$ be a real vector bundle. Choosing a metric on F, the complexification $F^c = F \underset{\mathbb{R}}{\otimes} \mathbb{C}$ acquires a holomorphic structure and therefore, there is a $\bar{\partial}$-operator

$$\bar{\partial}_{L_0}(F) \colon \Omega^0(\Sigma; L_0 \otimes F^c) \to \Omega^{0,1}(\Sigma; L_0 \otimes F^c).$$

Atiyah [2] shows that the function

$$\delta_{L_0}(F) = \dim_{\mathbb{C}} \ker(\bar{\partial}_{L_0}(F)) \mod 2$$

is independent of the choice of the metric, and that it extends to give a group homomorphism

$$\delta_{L_0} \colon KO(\Sigma) \to \mathbb{Z}/2.$$

Define $\gamma \in \widetilde{KO}(\Sigma)$ to be the pull-back of the generator of $\widetilde{KO}(S^2)$ under a map $\tilde{\Sigma} \to S^2$ of degree 1. Atiyah [2, Lemma (2.3)] shows that

$$\delta_{L_0}(\gamma) = 1.$$

Furthermore, the total Stiefel-Whitney class gives an isomorphism

$$w \colon \widetilde{KO}(\Sigma) \to \{1\} \oplus H^1(\Sigma; \mathbb{Z}/2) \oplus H^2(\Sigma; \mathbb{Z}/2)$$

of the additive group $\widetilde{KO}(\Sigma)$ onto the multiplicative group of the cohomology ring $H^*(\Sigma; \mathbb{Z}/2)$ (see [2, Remark, p. 54]). Clearly,

$$w(\gamma) = (1, 0, 1),$$

where we identify $H^2(\Sigma; \mathbb{Z}/2) = \mathbb{Z}/2$. We may, therefore, think of δ_{L_0} as a homomorphism of the multiplicative group of $H^*(\Sigma; \mathbb{Z}/2)$ to $\mathbb{Z}/2$, which takes the value 1 on the element $(1, 0, 1)$. Let $u \in H^1(\Sigma; \mathbb{Z}/2)$; then,

$$(1, u, 0) = (1, u, 1) \cdot (1, 0, 1)$$

in $H^*(\Sigma; \mathbb{Z}/2)$. Therefore,

$$\delta_{L_0}(1, u, 0) = \delta_{L_0}(1, u, 1) + 1. \tag{4.16}$$

Returning to (W, C) with $W = \pi_* M$ for $M \in \ker(1 + \tau^*)$, we see that

$$\delta(M) = \delta_{L_0}(W^r),$$

where W^r is a real rank two bundle, whose complexification is W. It follows from (4.16), that δ takes different values for different values of $w_2(W)$ and hence, that $w_2(W)$ determines whether M lies in P^+ or P^-.

From this discussion, we obtain the following explicit description of the subvariety $N_u^v \subset \mathbf{M}_u^v$ of local minima of f.

Proposition 4.23. *Let $u \in H^1(\Sigma; \mathbb{Z}/2) - \{0\}$, let $v \in H^2(\Sigma; \mathbb{Z}/2) = \mathbb{Z}/2$ and let P^+ and P^- be the Abelian varieties associated to the double cover of Σ, given by u as above. Then, the subvariety $N_u^v \subset \mathbf{M}_u^v$ of local minima of f is equal to P^+ and P^-, respectively, for the two values of v.*

Consequently, N_u^v is connected and, from the properness of f, it follows that $\mathbf{M}_u^v$ is connected, proving Theorem 4.22.

4.2.7 The $0 < d < 2g - 2$ case

In this subsection, we discuss two possible approaches to a proof of the following:

Conjecture 4.24. *The subspaces $\mathcal{M}_d \subset \mathcal{M}_{\mathrm{Sp}(4,\mathbb{R})}$ are connected for $0 < d < 2g - 2$.*

The Spectral Curve Approach

Let $N_d \subset \mathcal{M}_d$ be the subvariety of local minima. From Proposition 4.8, it follows that any point of N_d is of type $(2,2)$. Thus, N_d is the moduli space of pairs (V,c), where $\deg(V)=d$ and c is a K-valued quadratic form on V, such that (V,c) is poly-stable in the sense of Proposition 4.10. Note that $c \neq 0$; this can most easily be seen from stability of (E,Φ) but follows, of course, also from stability of (V,c).

Let $N'_d \subset N_d$ be the subvariety of pairs (V,c) for which $\mathrm{rk}(c)=2$ generically, and let $S = N_d - N'_d$ be the subvariety of pairs for which $\mathrm{rk}(c)=1$. Recall, from Remark 4.2, that $S=\varnothing$ for $d > g-1$. Let $(V,c)\in S$. There is a linebundle $L \subset V$ which is generically the kernel of c and thus, V is an extension

$$0 \to L \to V \to M \to 0,$$

where $M = L^{-1}\Lambda^2 V$. Clearly, c maps V to $M^{-1}\otimes K$ and it is determined by the induced section $\tilde{c} \in H^0(\Sigma; M^{-2}K)$. One can prove that the extension is non-trivial if and only if (V,c) is stable. Using this description, a dimension count shows that the codimension of S in N_d is at least 1. Thus, it suffices to prove that N'_d is connected.

For any pair (V,c) in N'_d, we can consider the associated spectral curve $\tilde{\Sigma} \to \Sigma$. However, we only know that this is smooth when $\det(c)$ has simple zeros (see Subsection 4.2.4). Consider the Hitchin map

$$\begin{aligned} H\colon N'_d &\to S^{4g-4-2d}(\Sigma) \\ (V,c) &\mapsto (\det(c)). \end{aligned}$$

Let $C^{4g-4-2d}(\Sigma) \subset S^{4g-4-2d}(\Sigma)$ be the space of effective divisors without multiple points, and let $D \in C^{4g-4-2d}(\Sigma)$. Then, the spectral curve of any $(V,c)\in H^{-1}(D)$ is the ramified double cover $\tilde{\Sigma}$ of Σ, with ramification divisor D. Using the theory of the Prym variety (see Mumford [25]) and Propositions 4.14 and 4.15, one can prove that $H^{-1}(D)$ is isomorphic to the Prym variety of the cover. The Prym variety is an Abelian variety of dimension $3g-3-d$ and it follows that the subspace $H^{-1}(C^{4g-4-2d}(\Sigma))$ of N'_d is connected.

Remark 4.25. As a consequence of this description, we see that $N_d \neq \varnothing$ for $0 < d < 2g-2$.

One way of turning this into a proof, would be to show that a stable pair (V,C), where $\det(c)$ has multiple zeros, can be deformed into one for which $\det(c)$ has simple zeros. Another way could be to investigate further the correspondence between rank 1 torsion free sheaves on the spectral curve and Higgs bundles on Σ, when the spectral curve is not smooth.

The Direct Approach

Consider (V, c) as an ordinary Higgs bundle, as described in Subsection 4.2.4. Here, we think of c as lying in

$$H^0(\Sigma; \mathrm{End}_0(V) \otimes L),$$

where $L = \Lambda^2 V^* K$. We can then show that stability of (V, c) in the sense of Proposition 4.10 implies stability of (V, c) as a Higgs bundle. However, the converse is not true: (V, c) must satisfy three further conditions:

i) For *any* subbundle M of V, $\deg(M) < \deg(V)$.

ii) When $\deg(V) > 0$, c must be non-zero.

iii) When $\mathrm{rk}(c) = 1$, let $U \subset V$ be the subbundle which is generically the kernel of c, and let $U' \subset V$ be the subbundle such that $U'L$ is generically the image of c. Then, U and U' must have negative degree.

Nitsure [27] shows that the moduli space of stable Higgs bundles (V, c) is connected. Letting U be the subspace of stable Higgs bundles (V, c), which satisfy the conditions above, it follows that there is surjective map from U to N_d. The problem is to show that U is connected.

4.3 Fibrations

In this section, we briefly discuss a possible application of our results.

Let X be a 4-manifold, which is the total space of a fibre bundle

$$X \xrightarrow{\pi} \Sigma,$$

over a surface of genus $g \geqslant 2$, such that the fibre S_x over $x \in \Sigma$ is a surface of genus h. The fundamental group $\pi_1\Sigma$ acts on the cohomology $H^1(S_x;\mathbb{R})$ preserving the non-degenerate skew form given by the cup product. This way, we obtain an associated flat $\mathrm{Sp}(2h,\mathbb{R})$-bundle

$$E = \tilde{\Sigma} \underset{\pi_1\Sigma}{\times} H^1(S_x;\mathbb{R})$$

over Σ. Here, $\tilde{\Sigma}$ is the universal cover of Σ.

This idea goes back to Griffiths [17], who considered families of varieties. Taking the Dolbeault cohomology of the fibres, led him to what he calls a variation of Hodge structure. This, in turn, was the inspiration for Simpson [28], who realised that the variations of Hodge structure are examples of Higgs bundles.

We get an invariant of the fibration by taking the degree d of the bundle V, obtained by a reduction to the maximal compact subgroup $\mathrm{U}(h)$. When the flat connection is reductive and $d = h(g-1)$, we obtain a further invariant by taking the Stiefel-Whitney classes w_1 and w_2 of the $\mathrm{O}(h,\mathbb{C})$ bundle W obtained from V by Theorem 4.3.

A slightly more general situation was considered by Atiyah [1]. He proves the following theorem.

Theorem 4.26 (Atiyah [1]). *The number d is determined by the signature of X as follows:*

$$\mathrm{Sign}(X) = 4d.$$

One can now ask the following questions in the case $d = h(g-1)$:

1. Do the further invariants given by the Stiefel-Whitney classes occur? We have seen in the case of $h = 2$, that flat $\mathrm{Sp}(4,\mathbb{R})$-bundles with all the values given by Theorem 4.4 do occur. However, we do not know whether they can be induced by a fibration as above.

2. If these invariants do occur, are they then invariants of the fibration only, or of the 4-manifold X? It follows from Atiyah's theorem that d is an invariant of X, but is this also the case for w_1 and w_2? This is perhaps not so likely for w_1, as it is defined in terms of the cohomology of Σ.

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