An Overview of IFS with Applications to Thermodynamic Formalism

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Outline

IFS theory

Iterated Function Systems (IFS) were originally introduced as a theoretical framework for the study and generation of fractal sets and their applications. Although significant developments had already taken place in previous decades–notably Mandelbrot's work on fractals arising from complex functions and other geometrically constructed sets–it was not until the 1980s that the pioneering contributions of J. Hutchinson and M. Barnsley established the theoretical foundations of what is now known as the Hutchinson–Barnsley theory. This framework rigorously demonstrates the existence of a compact attractor set and an invariant measure supported on that attractor.

Dynamical Systems defined by IFS

Typically, an IFS is a family of self-maps $f_j : X \to X, \quad j \in J$, defined on a set X (usually a metric or topological space). We denote such a system by $R = (X, f_j)_{j \in J}$.
For a single dynamical system $T : X \to X$, each point $x \in X$ generates a unique orbit $x,\ T(x),\ T^2(x),\ \ldots,$ whose long-term behavior can be analyzed through asymptotic notions such as the ω-limit set, the non-wandering set, or recurrence.
In contrast, for an IFS, the iteration process is *non-autonomous*. To iterate a point $x \in X$, one must choose a sequence $(j_1, j_2, \ldots) \in J^{\mathbb{N}}$ and compute $x,\ f_{j_1}(x),\ f_{j_2}(f_{j_1}(x)),\ \ldots$ Thus, where does the asymptotic behavior reside in this non-autonomous setting?

Dynamical Systems defined by IFS

Actually, the only points which are dynamically significant in X are the cluster points of compositions

$$f_{i_1}(f_{i_2}(\cdots(f_{i_k}(x))))$$

for $i_1, i_2, ..., i_k \in J$ and k arbitrary large!
To capture that special points J. Hutchinson introduced the fractal operator $F : K^*(X) \to K^*(X)$ given by

$$F(B) := \bigcup_{j \in J} f_j(B), \; B \in K^*(X).$$

It happens that $y_k := f_{i_1}(f_{i_2}(\cdots(f_{i_k}(x)))), x \in B$ is precisely an element of $F^k(B)$.

Dynamical Systems defined by IFS

The geniality of Hutchinson's approach lies in considering a classical dynamical system $F : K^*(X) \to K^*(X)$, whose ω-limit set coincides with the union of the ω-limits of all orbits generated by the IFS.

Under a contractivity assumption, Hutchinson demonstrated that F is a contraction with respect to the Hausdorff distance. Consequently, the ω-limit of F is a unique fixed point $A \in K^*(X)$, referred to as the **attractor of the IFS**. This attractor can be obtained by iterating F on any nonempty compact set B:

$$A = \lim_{k\to\infty} F^k(B).$$

This result establishes the first part of the Hutchinson–Barnsley theory for contractive IFSs.

Dynamical Systems defined by IFS

It is straightforward to see that

$$A = \lim_{k \to \infty} F^k(X) = \bigcap_{k \geqslant 0} F^k(X)$$

provides an alternative representation of the attractor set. We will not discuss several technical topics, such as the code space, code map, and semi-conjugation via skew maps. Nevertheless, these concepts constitute powerful tools for the study of IFS. Briefly, we define the code map

$$\pi(i) = \lim_{k \to \infty} f_{i_1}\big(f_{i_2}(\cdots (f_{i_k}(X))\big),$$

where $i = (i_1, i_2, \ldots) \in \Sigma = J^{\mathbb{N}}$ (the full shift space).

Dynamical Systems defined by IFS

One can then construct a skew-product map $S : X \times \Sigma \to X \times \Sigma$ defined by

$$S(x, i) = (f_{i_1}(x), \sigma(i)),$$

where $\sigma(i) = (i_2, i_3, \ldots)$ denotes the one-sided shift map. This defines a new autonomous dynamical system whose projection onto the first coordinate of S^k yields the orbits of the original IFS. For example:

$$S^3(x, i) = (f_{i_3}(f_{i_2}(f_{i_1}(x))), (i_4, i_5, \ldots)).$$

Moreover, it can be shown that the original IFS is semi-conjugate to a standard IFS on the shift space.

Dynamical Systems defined by IFS

Classical examples are the middle-third Cantor set, which is the attractor of the IFS: Consider $J = \{1, 2\}$, $X = [0, 1]$ and the functions $\phi_j : X \to X$ given by

$$\begin{cases} \phi_1(x) = \frac{x}{3} \\ \phi_2(x) = \frac{x}{3} + \frac{2}{3}. \end{cases}$$

Dynamical Systems defined by IFS

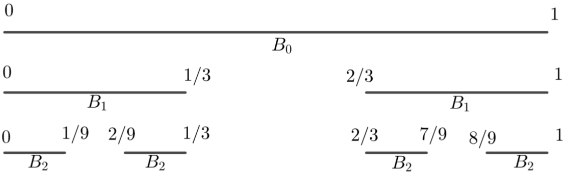


Figure: Construction of the middle-third Cantor set: $B_0 = [0, 1]$, $B_1 = F([0, 1])$,$B_2 = F^2([0, 1])$, etc.

Dynamical Systems defined by IFS

We also have the Sierpinski triangle S, which is the attractor of the IFS where $X = T$ is an equilateral triangle with vertices $(0,0)$, $(1,0)$ and $(1/2, \sqrt{3}/2)$, $J = \{1,2,3\}$ and the IFS given by the functions $\phi_j : T \to T$ given by

$$\begin{cases} \phi_1(x,y) = (\frac{x}{2}, \frac{y}{2}) \\ \phi_2(x,y) = (\frac{x}{2} + \frac{1}{2}, \frac{y}{2}) \\ \phi_3(x,y) = (\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}). \end{cases}$$

Dynamical Systems defined by IFS

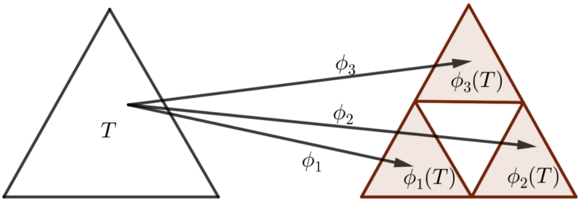


Figure: Iterate $n = 1$, $F(T) = \phi_1(T) \cup \phi_2(T) \cup \phi_3(T)$ for the Sierpinski set S

Dynamical Systems defined by IFS

The iterates $F^n(T)$ are:

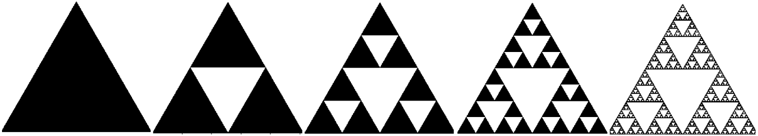


Figure: Iterates $n = 0, 1, 2, 3$ and 10, of F, approximating the Sierpinski set S

Stochastic view of IFS

For a dynamical system $T : X \to X$, where each point $x \in X$ generates a single orbit $(x, T(x), T^2(x), \ldots)$, the stochastic aspects of the dynamics are studied within the framework of ergodic theory. In this context, one seeks invariant measures, that is, probability measures μ satisfying

$$\mu(B) = \mu(T^{-1}(B)) \quad \forall\, B \subset X.$$

Such measures encode the long-term statistical behavior of orbits through the Birkhoff Ergodic Theorem.
The natural question that arises is how to extend this framework to the case of an IFS.

Stochastic view of IFS

For a contractive IFS the orbit asymptotic behavior is independent of the initial point x (here lies a big difference from ergodic theory):

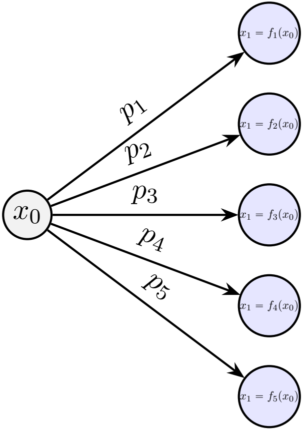
$$x, f_{j_1}(x), f_{j_2}(f_{j_1}(x)), \ldots,$$

it depends strongly on the choices $j_1, j_2, \ldots$. Therefore, the natural way to introduce randomness is to put a fixed probability for each symbol

$$\mathbb{P}(j_k = j) = p_j \in [0, 1],$$

where $\sum_j p_j = 1$.

Stochastic view of IFS



Stochastic view of IFS

For a contractive IFS, J. Hutchinson introduced the notion of an IFS with probabilities (IFSp). An IFSp is an IFS $R = (X, f_j)_{j\in J}$ endowed with a family of probabilities $(p_j)_{j\in J}$, where each symbol $j \in J$ is chosen at each iteration with probability p_j. Thus, one writes $R = (X, f_j, p_j)_{j\in J}$.
By analogy with the invariance equation $\mu(B) = \mu(T^{-1}(B))$, one defines the **Markov operator** $M : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$M(\mu)(B) = \sum_{j\in J} p_j\, \mu(f_j^{-1}(B)), \quad \forall B \subset X.$$

Stochastic view of IFS

Given an initial distribution μ on X, the sequence

$$\mu,\ M(\mu),\ M^2(\mu),\ \dots$$

generates a stochastic process whose distribution at time k corresponds to that of

$$f_{i_1}\big(f_{i_2}(\cdots(f_{i_k}(x)))\big),$$

where the indices $i_1, i_2, \dots, i_k \in J$ are independently chosen according to the respective probabilities p_j.

Stochastic view of IFS

In his seminal article, J. Hutchinson proved that for an IFSp $R = (X, f_j, p_j)_{j \in J}$, the Markov operator $M : \mathcal{P}(X) \to \mathcal{P}(X)$ is a contraction in the space of probability measures with respect to the Monge-Kantorovich distance. Consequently, for any initial distribution μ, the stochastic process $\mu, M(\mu), M^2(\mu), \ldots$ converges to a unique probability measure ν satisfying $M(\nu) = \nu$ (stationary), which is referred to as the **invariant measure for the IFSp** (or Hutchinson measure).

Stochastic view of IFS

Moreover, assuming $p_j > 0$ for all j, one can show that

$$\mathrm{supp}(\nu) = A.$$

This completes the second part of the theory, since the stationary distribution ν assigns positive probability to every dynamically significant point, i.e., to all points in the attractor.

Stochastic view of IFS

We say that an IFSp $R = (X, f_j, p_j)_{j \in J}$ satisfy the Hutchinson–Barnsley theory if the following theorem is valid

Theorem 1 (folklore)

1. *For any $B \in K^*(X)$ the sequence $F^k(B), k \geqslant 0$ converges a single set$A \in K^*(X)$ (the attractor set) satisfying $F(A) = A$, where $F : K^*(X) \to K^*(X)$ is the fractal operator;*
2. *For any initial distribution μ the stochastic process $\mu, M(\mu), M^2(\mu), \ldots$ converges to a unique probability ν satisfying $M(\nu) = \nu$ (the invariant measure), where $M : \mathcal{P}(X) \to \mathcal{P}(X)$;*
3. $\operatorname{supp}(\nu) = A$.

Since the 1980s, considerable effort has been devoted to extending this theorem to various levels of generality.

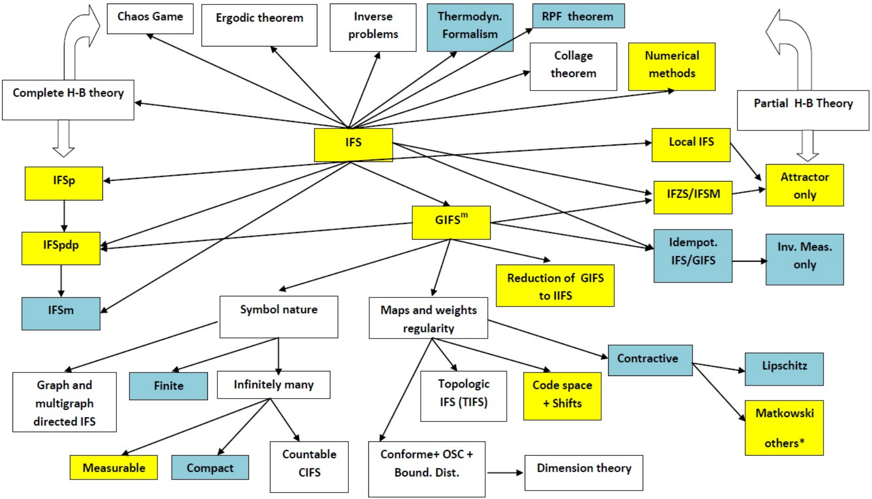
The first formulation

The first formulation of the Hutchinson–Barnsley theory was introduced by J. Hutchinson in the article *Fractals and Self-Similarity* (1981). In this work, (X, d) denotes a complete metric space, $J = \{1, \dots, n\}$ represents a finite set of maps, each f_j is a contraction, and $p_j \in (0, 1)$ are finite, positive, and constant probabilities. Over the past decades, the following parameters have been generalized:

Generalizations overview

1. The space X may be metric/topological, complete/compact; one may also consider powers of X with maps $f_j : X^m \to X$, as well as variations of metric structures.
2. The regularity of each map f_j, which may include generalized contractions, weakly hyperbolic maps, or contractions enriched with non-expansive maps.
3. The space of symbols J, which may be countable, possibly infinite, measurable, or compact infinite.
4. The nature of the probabilities, which may be constant or dependent on the location; in the latter case, the regularity of this dependence must be considered.
5. The type of set theory employed, whether crisp or fuzzy.
6. The nature of the algebra, either the standard algebra $(\mathbb{R}, +, \cdot)$ or the max-plus (tropical) algebra $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, including possible variations.
7. The relational properties and domains of the maps, including local IFSs and (multi)graph-directed IFSs.

IFS Theories Diagram



Recent developments

We now present selected results from joint work with our research group, primarily in the continuous setting, under the assumption of certain contractivity conditions and IFSs with a compact set of maps.

The Hutchinson–Barnsley theory for IFSm

The results here are related to the paper [**OS26**], *The Hutchinson-Barnsley theory for IFSs with general measures* which is a collaboration with prof. Rafael R. Souza, to appear in JMAA.

Definition 2

An IFS is a map $\tau : \Lambda \times X \to X$ which is continuous.
Sometimes we denote the maps as $\tau(\lambda, x) = \tau_\lambda(x)$, to simplify the notation.

The Hutchinson–Barnsley theory for IFSm

Definition 3

For each IFS $\mathcal{R} = (X, \tau)$ we define the fractal operator by $F_{\mathcal{R}} : \mathcal{K}^*(X) \to \mathcal{K}^*(X)$ by

$$F_{\mathcal{R}}(B) := \tau(\Lambda \times B) = \cup_{\lambda \in \Lambda} \tau_\lambda(B),$$

for any $B \in \mathcal{K}^*(X)$.

The Hutchinson–Barnsley theory for IFSm

Definition 4

An IFS with measures, IFSm for short, is an IFS $\tau : \Lambda \times X \to X$ endowed with a continuous family of measures $q : X \to \mathcal{P}(\Lambda)$.

We denote any IFSm by $\mathcal{R} = (X, \tau, q)$ (compact notation), where $\lambda \in \Lambda$ and $q : X \to \mathcal{P}(\Lambda)$.

The Hutchinson–Barnsley theory for IFSm

Definition 5

To each IFSm we assign a transfer operator $B_q : C(X) \to C(X)$ defined by

$$B_q(f)(x) := \int_{\Lambda} f(\tau(\lambda, x)) dq_x(\lambda) \tag{1}$$

for any $f \in C(X)$.

The Hutchinson–Barnsley theory for IFSm

By duality, B_q induces a new operator in $\mathcal{P}(X)$:

Definition 6

To each IFSm we assign a Markov operator $T_q := B_q^* : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by duality, for any $\mu \in \mathcal{P}(X)$ by

$$\int_X f(x) dT_q(\mu) := \int_X B_q(f)(x) d\mu(x) := \int_X \int_\Lambda f(\tau(\lambda, x)) dq_x(\lambda) d\mu(x) \tag{2}$$

for any $f \in C(X)$. A probability $\mu \in \mathcal{P}(X)$ is called invariant (with respect to the IFSm) if $T_q(\mu) = \mu$.

The Hutchinson–Barnsley theory for IFSm

Theorem 7

Let $\mathcal{R} = (X, \tau, q)$ be a normalized IFSm, under mild assumptions there exists a unique compact set $A_{\mathcal{R}}$ called fractal attractor and a unique probability $\mu_{\mathcal{R}}$ called invariant or Hutchinson measure, such that

1. $F_{\mathcal{R}}(A_{\mathcal{R}}) = A_{\mathcal{R}}$ *and for any nonempty compact set* $B \subseteq X$ *we have* $F_{\mathcal{R}}^k(B) \to A_{\mathcal{R}}$*, w.r.t. the Hausdorff distance;*
2. $T_q(\mu_{\mathcal{R}}) = \mu_{\mathcal{R}}$ *and for any probability* $\nu \in \mathcal{P}(X)$ *we have* $T_q^k(\nu) \to \mu_{\mathcal{R}}$*, w.r.t. the Monge–Kantorovich distance;*
3. $\mathrm{supp}(\mu_{\mathcal{R}}) = A_{\mathcal{R}}$.

The Hutchinson–Barnsley theory for IFSm

In the paper we provide several sets of conditions regarding the joint regularity of τ_λ and q_x, including non expansive and eventually contractive settings:

- M1. there exists $s > 0$ such that, for any pair x and y in X, we have $\int_\Lambda |f(\tau(\lambda, x)) - f(\tau(\lambda, y))|\, dq_x(\lambda) < s\, d(x, y)$ for any map $f \in Lip_1(X)$;
- H2. there exists $r \geqslant 0$, such that, the map $\tau(\cdot, x) \in \mathrm{Lip}_r(\Lambda)$, uniformly with respect to x;
- H3. there exists $t \geqslant 0$, such that, the map $q : X \to \mathcal{P}(\Lambda)$ is in $\mathrm{Lip}_t(X)$;
- H4. $q_x(A) > 0$ for any open subset $A \subseteq \Lambda$ and $x \in X$.

The Hutchinson–Barnsley theory for IFSm

Note that M1 holds for example if the following stronger assumption is assumed:

C1. the map $\tau(\lambda, \cdot) \in \mathrm{Lip}_s(X)$, with $s < 1$, uniformly with respect to λ.

On the other side, the following assumption is weaker that C1, and will be needed for some of our further results:

W1. the map $\tau(\lambda, \cdot) \in \mathrm{Lip}_1(X)$, uniformly with respect to λ.

The Hutchinson–Barnsley theory for IFSm

The next theorem can be found in Lewellen 1993 (also Mendivil 1998) but we provide a different and simpler proof.

Theorem 8

Consider an IFSm $\mathcal{R} := (X, \tau)$ satisfying C1. Then, $F_{\mathcal{R}}$ is s-Lipschitz contractive. In particular, there exists a unique invariant set $A_{\mathcal{R}}$.

The Hutchinson–Barnsley theory for IFSm

From Mendivil 1998 we know that in the case of IFSm where the family of measures $q : X \to \mathcal{P}(\Lambda)$ is constant, then there exists a unique invariant probability $\mu_{\mathcal{R}}$ for the IFSm (also from Arbieto, Junqueira and Santiago 2016 for weakly hyperbolic IFS, but still for a constant q_x). We will now extend this result in fairly greater generality:

Theorem 9

Consider an IFSm $\mathcal{R} := (X, \tau, q)$ satisfying the conditions M1, H2 and H3 above. Then T_q is $s + r \cdot t$-Lipschitz.

So, if $s + r \cdot t < 1$ then the existence of the invariant measure, and consequently the second part of Theorem 7 holds via Banach contraction theorem.

The Hutchinson–Barnsley theory for IFSm

Some of our results will remain valid if the uniform contraction holds for composition of maps: We have the following assumption, weaker than C1:

CP1. there is $M \geqslant 1$ and $0 < s < 1$ such that the map $\tau_{\lambda^M}(\cdot) \in \mathrm{Lip}_s(X)$, uniformly with respect to λ^M.

The Hutchinson–Barnsley theory for IFSm

The same conclusion in the next theorem holds under the stronger (but simpler) hypotheses C1, H2, H3, H4. However we choose to prove that under less restrictive conditions.

Theorem 10

Under hypothesis W1, CP1, H2, H3 and H4, we have

$$\mathrm{supp}(\mu_{\mathcal{R}}) = A_{\mathcal{R}}.$$

This result conclude the proof of Theorem 7.

The Hutchinson–Barnsley theory for IFSm

Assume (X, d) a compact metric space, (X, τ) a uniformly contractible IFS, and the family of measures is given by

$$dq_x(\lambda) = e^{A(\lambda,x)} d\nu(\lambda)$$

for a Lipschitz potential $A : \Lambda \times X \to \mathbb{R}$ and a fixed probability $\nu \in \mathcal{P}(\Lambda)$ whose support is Λ. From [**MO25**] (also from [**LMMS15**] for the Hölder version) we can always find $\tilde{A} : \Lambda \times X \to \mathbb{R}$ cohomologous to A (i.e., there exists a continuous function h and a constant c such that $\tilde{A}(\lambda, x) = A(\lambda, x) + h(\tau_\lambda(x)) - h(x) - c)$), also Lipschitz continuous, such that $x \to \int_\Lambda e^{\tilde{A}(\lambda,x)} d\nu(\lambda) \equiv 1$. Thus, q_x can always be assumed to be a probability for any $x \in X$. One can show that the condition H3 is satisfied for

$$t := \mathrm{diam}(\Lambda) e^{\|\tilde{A}\|_\infty} \mathrm{Lip}(\tilde{A})\ s$$

The thermodynamic formalism problem

The thermodynamic formalism was introduced by D. Ruelle in 1967 ([**Rue67**]). The modern formulation – expressed in terms of symbolic dynamical systems and building on earlier work of P. Walters on g-measures in the 1970s – was established by Parry and Pollicott in 1990 ([**PP90**]). See also Fan and Lau [**FL99**] for a first application to place dependent IFSp.

Holonomic entropy and pressure

We now present some results from [**BOS23**]. One the holonomic probabilities associated with an IFS. These are measures that satisfy the variational principle for pressure with respect to the variational entropy. More precisely, given an IFS $(X, (\phi_j)_{j\in J})$, one defines the set of holonomic probabilities by

$$\mathcal{H} := \left\{ \Pi \in \mathcal{P}(X \times J) \;\middle|\; \int f(\phi_j(x)) - f(x)\, d\Pi(x, j) = 0 \right\}$$

where $f \in C(X, \mathbb{R})$.

Holonomic entropy and pressure

The transfer operator B_q is given by

$$B_q(f)(x) = \int_J f(\phi_j(x))\, dq_x(j), \quad \forall x \in X.$$

and the pressure function

$$P(A) = \sup_{\Pi \in \mathcal{H}} \inf_{\substack{f \in C(X;\mathbb{R}) \\ f>0}} \left\{ \int_X \ln \frac{B_q(f)}{f}\, d\Pi \right\},$$

where, $A > 0$, $\mu \in \mathcal{P}(J)$ and $dq_x(\theta) = A(\phi_j(x))\, d\mu(j)$, satisfies the following variational principle (proved in [**BOS23**]):

Holonomic entropy and pressure

$$P(A) = \sup_{\Pi \in \mathcal{H}} \left\{ h_v^{\mu}(\Pi) + \int_X \log A \, d\Pi \right\},$$

where the variational entropy of Π with respect to the a priori probability μ is defined as

$$h_v^{\mu}(\Pi) := \inf_{\substack{f \in C(X,\mathbb{R}) \\ f > 0}} \left\{ \int_X \ln \frac{B_\mu(f)}{f} \, d\Pi \right\},$$

where

$$B_\mu(f)(x) = \int_J f(\phi_j(x)) \, d\mu(j), \quad \forall x \in X.$$

Equilibrium states

One can prove that the variational entropy is usc obtaining, as consequence of it, that:

Theorem 11 (Existence of Equilibrium States)

Let $\mathcal{R}$ be an IFSm, $\psi : X \to \mathbb{R}$ a positive continuous function and μ a probability on Θ. Then, the set of equilibrium states for (ψ, μ) is not empty.

Pressure differentiability

One can show a uniqueness result for the equilibrium state. In order to do that we will need to consider the functional $p : C(X, \mathbb{R}) \to \mathbb{R}$, which $A = \exp(\varphi)$, given by

$$p(\varphi) = P(\exp(\varphi)). \tag{3}$$

It is immediate to verify that p is a convex and finite valued functional. We say that a Borel signed measure $\nu \in \mathcal{M}_s(X)$ is a **sub-gradient** of p at φ if it satisfies the following sub-gradient inequality

$$p(\eta) \geqslant p(\varphi) + \nu(\eta - \varphi) \text{ for any } \eta \in \mathcal{M}_s(X).$$

The set of all sub-gradients at φ is called **sub-differential** of p at φ and denoted by $\partial p(\varphi)$.

Pressure differentiability

Theorem 12 (Pressure Sub-differentiability)

For any fixed $\varphi \in C(X,\mathbb{R})$ we have

1. *the signed measure $\nu \in \partial p(\varphi)$ if, and only if, $\nu(\eta) \leqslant d^{+}p(\varphi)(\eta)$ for all $\eta \in C(X,\mathbb{R})$;*
2. *the set $\partial p(\varphi)$ is a singleton if, and only if, $d^{+}p(\varphi)$ is the Gâteaux derivative of p at φ.*

Uniqueness of equilibrium states

Corollary 13 (Uniqueness of equilibrium states)

Let $\mathcal{R}$ be an IFS, $\psi : \boldsymbol{X} \to \mathbb{R}$ a positive continuous function, μ a probability on Θ and $\Phi(\hat{\nu}) = \nu$ (the projection push-forward map) for $\hat{\nu} \in \mathcal{H}(\mathcal{R})$, where ν is given by disintegration with respect to π. If the functional p defined is Gâteaux differentiable at $\varphi \equiv \log \psi$, then

$$\#\{\Phi(\hat{\mu}) : \hat{\mu} \textit{ is an equilibrium state for } \psi\} = 1.$$

The proof consists in proving that for any equilibrium state $\hat{\mu}$ for ψ we have $\Phi(\hat{\mu}) = \partial p(\varphi)$.

Final remarks

Some open questions are:

1. IS there a RPF theorem for the (non-normalized)IFSm setting, or one can at least prove something like [**MO25**]? If yes, we could improve [**BOS23**] and apply [**OS26**].
2. In [**OS26**] we prove the Hutchinson-Barnsley theory of a normalized IFSm. Is the invariant probability ergodic (Elton 86, Forte 93, Arbieto 17)?
3. What could be more general than an IFSm? Are there any applications?
4. Can the H-B theory for mpIFS and IFZS be completed?

Final remarks

The theory of idempotent IFSs, more precisely the max-plus algebra has been successfully used to study Thermodynamic formalism. In the setting of expanding transformations , the preprint [**ZY25**], "Tropical thermodynamic formalism" by Zhiqiang and Sun provides a good example. They investigate the zero-temperature large deviation principle for equilibrium states in the context of distance-expanding maps. The logarithmic-type zero-temperature limit in the large deviation principle induces a tropical algebra structure, which motivates our study of the tropical adjoint Bousch operator $\mathcal{L}_A^*$ since the Bousch operator $\mathcal{L}_A$ is tropical linear and corresponds to the Ruelle operator $\mathcal{R}_A$.

Final remarks

The Bousch operator $\mathcal{L}_A(g)(x) = \max_{T(y)=x} A(y) + g(y)$ has a tropical or max-plus eigen value/function (m, V) (sometimes called sub-action) satisfying $\mathcal{L}_A(V)(x) = m+V(x)$. Hence, equivalent to

$$\max_{T(y)=x} A(y) + V(y) - V(T(y)) = m,$$

where $m = \max_{T(\mu)=\mu} \int_X A d\mu$. This is the ergodic optimization analogous to the Hamilton-Jacobi equation in the Aubry-Mather theory. If we consider the IFS formed by the contracting inverse branches ϕ_j of the expanding map T the equation becomes

$$\max_{j\in J} A(\phi_j(x)) + V(\phi_j(x)) - V(x) = m,$$

and shows why holonomic measures are important in IFS ergodic optimization.

Final remarks

Also, in the joint work [**MO25**] we consider two compact metric spaces J and X and a uniform contractible IFS $\{\phi_j : X \to X \mid j \in J\}$. For a Lipschitz continuous function A on $J \times X$ and for each $\beta > 0$ we consider the Gibbs probability $\rho_{_{\beta A}}$. The goal is to study a large deviation principle for such family of probabilities as $\beta \to +\infty$ and its connections with idempotent probabilities. In the non-place dependent case ($A(j,x) = A_j, \forall x \in X$) we will prove that $(\rho_{_{\beta A}})$ satisfy a LDP and $-I$ (where I is the deviation function) is the density of the unique invariant idempotent probability for a mpIFS associated to A. In the place dependent case, we prove that, if $(\rho_{_{\beta A}})$ satisfy a LDP, then $-I$ is the density of an invariant idempotent probability.

Final remarks

Regarding the classical level-2 thermodynamic formalism, [**LO24**] study the dynamical system given by the push-forward map $\mathfrak{T} : \mathcal{M} \to \mathcal{M}$ on the space $\mathcal{M}$ of Borel probabilities over $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$. For a continuous potential $A : \mathcal{M} \to \mathbb{R}$, an a priori measure Π_0, and a convolution operation on $\mathcal{M}$, one define a Level-2 IFS Ruelle operator and prove the existence of an eigenfunction and an eigenprobability $\hat{\Pi} \in \mathfrak{M}$. Under normalization of A, one obtain $\mathfrak{T}$-invariant probabilities and define their variational entropy and pressure.
Examples show how Level-2 eigenprobabilities in $\mathfrak{M}$ naturally extend classical Level-1 Thermodynamic Formalism results.

Final remarks

Finally, in the recent preprint [**LMO26**], we demonstrate how idempotent probabilities can be employed to define level-2 idempotent pressure functions and entropy, thereby opening new avenues for applications and revealing novel phenomena such as phase transitions, non-uniqueness of equilibrium states, and more.

Final remarks

Theorem 14 (LMO26)

If $\ell : C(\mathcal{P}(X), \mathbb{R}) \to \mathbb{R}$ is an idempotent pressure function, then there exists a unique upper semi-continuous (u.s.c.) function $h : \mathcal{P}(X) \to \mathbb{R}_{\max}$ such that

$$\ell(g) = \sup_{\mu \in \mathcal{P}(X)} [h(\mu) + g(\mu)], \tag{4}$$

for any $g \in C(\mathcal{P}(X), \mathbb{R})$. Reciprocally, if $h : \mathcal{P}(X) \to \mathbb{R}_{\max}$ is bounded above, and it is not identically $-\infty$ then equation (4) defines an idempotent pressure function.

Final remarks

Definition 15

Let (X, d) be a compact metric space, $\ell : C(\mathcal{P}(X), \mathbb{R}) \to \mathbb{R}$ be an idempotent pressure and $h_\ell : \mathcal{P}(X) \to \mathbb{R}$ be the unique u.s.c. function satisfying (4). We say that h_ℓ is the **density entropy** associated to the idempotent pressure function ℓ. Moreover, given ℓ, we call any probability $\nu \in \mathcal{P}(X)$ attaining the supremum, that is,

$$\ell(g) = h_\ell(\nu) + g(\nu), \tag{5}$$

an **equilibrium state** associated to the idempotent pressure function ℓ.

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Obrigado!!
Merci!!
Danke!!
Gracias!!
Thank you!