# CMUP Informal PhD Seminar

2021/2022 - **Session 5**

Date: **Wednesday, January 12, 17h00**

Place: **Room 0.04** (Building FC1, FCUP)

Program:

**17h00 -** Pedro Ribeiro (PIUDM):

***On The Zeros of a Class of Dirichlet series***


**Abstract:**

Let $Q(x,y) = ax^2 + bxy + cy^2$ be a real and positive definite quadratic form. The classical Epstein zeta function is defined as the Dirichlet series

$$Z_2(s,\,Q) = \sum_{m,n\neq 0} \frac{1}{Q(m,n)^s}, \quad \text{Re}(s) > 1, \tag{1}$$

where the notation given in the subscript $m,\,n \neq 0$ means that only the term $m = n = 0$ is omitted from the infinite series.

In 1949, S. Chowla and A. Selberg announced the following formula for (1), valid in the entire complex plane,

$$a^s\Gamma(s)\,Z_2(s,\,Q) = 2\Gamma(s)\zeta(2s) + 2k^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) + 8k^{1/2-s}\pi^s\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\,\cos\left(n\pi b/a\right)K_{s-1/2}\left(2\pi k\,n\right), \tag{2}$$

where $d := b^2 - 4ac$ is the discriminant of the quadratic form, $k^2 := |d|/4a^2$ and $\sigma_\nu(n) = \sum_{d|n} d^\nu$ is the generalized divisor function of index $\nu$. Also, $\zeta(s)$ denotes the classical Riemann zeta function and $K_\nu$ the modified Bessel function.

By using no more than basic tools of Complex and Fourier analysis, in this talk we will discuss generalizations of (2) for large classes of Dirichlet series. In particular, we will see how the representation (2) is connected with the infinitude of zeros of $\zeta(s)$ at the critical line $\text{Re}(s) = \frac{1}{2}$ (Hardy's Theorem). As a very simple application of our method, we will argue that the conclusion of Hardy's Theorem for $\zeta(s)$ can be deduced from Jacobi's 4-square Theorem, which constitutes a curious connection between a purely analytic theorem involving deep properties of $\zeta(s)$ and an interesting arithmetical property concerning the representation of a given integer as the sum of four squares.


**17h50 -** Coffee Break