Representation dimension: overview and recent results

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6 de dezembro de 2018

Introduction

The use of homolgical techniques has been very useful in the representation theory of algebras. In general, we use homological invariants to *measure* how far an algebra or a module is from having *good properties*. This is the basic idea behind the concept of *representation dimension*.

Notations

- A is an algebra, that is, a finite dimensional algebra over a field k.
- $\text{mod}A$ is the category of finitely generated left A-modules, while $\text{ind}A$ is a subcategory containing the isoclasses of indecomposable A-modules.
- $\text{pd}M$ indicates the projective dimension of a module M and $\text{id}M$ its injective dimension.

The projective dimension measures, in a way, how close a module is from being projective. Dual remark for the injective dimension.

Global dimension

Definition

The global dimension of an algebra A is defined as:

$$\text{gl.dim}A = \sup\{\text{pd } M : M \text{ is } A\text{-module}\} \quad \leq \infty$$

Global dimension

$$\text{gl.dim}A = 0 \iff \text{ all modules are projectives}$$

$$\text{gl.dim}A \leq 1 \iff \text{ pd}M \leq 1, \quad \forall \text{ module } M$$

$$\iff \text{ each submodule of a projective is projective}$$

In this case, we say that A is hereditary.

The global dimension measures how far an algebra is from being hereditary.

Algebras of global dimension 2

The *tilting* process

A module T is tilting if

- $\operatorname{pd} T \leq 1$
- $\operatorname{Ext}^1_A(T, T) = 0$
- There exists a short exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with $T_i \in \operatorname{add} T$

Tilted algebras (Brenner-Butler e Happel-Ringel)

If A is a hereditary algebra and T is tilting, then $\mathrm{End}_A(T)$ is called a tilted algebra.

If A is tilted, then $\mathrm{gl.dim} A \leq 2$ and for each indecomposable module X, $\mathrm{pd} X \leq 1$ or $\mathrm{id} X \leq 1$.

Algebras defined from homological dimensions

Quasitilted algebras (Happel-Reiten-Smalø)

An algebra A is quasitilted if $\text{gl.dim} A \leq 2$ and for each indecomposable module X, $\text{pd} X \leq 1$ or $\text{id} X \leq 1$.

Shod algebras (C-Lanzilotta)

An algebra A is shod if for each indecomposable module X, $\text{pd} X \leq 1$ or $\text{id} X \leq 1$.
Shod algebras have global dimension at most 3.

Representation dimension

- The representation dimension of an algebra was introduced by Auslander in his famous Queen's Notes in the seventies ([Auslander, M., *Representation dimension of artin algebras*, Math. Notes, Queen Mary College, London (1971)]).
- Marked renewal of interest about fifteen years ago.
- But still: *What does this dimension mean ?*

Representantion dimension and Auslander generators

A generator-cogenerator for A is an A-module M such that

$$A \oplus \mathrm{D}A \in \mathrm{add}M.$$

Representation dimension of A

$$\text{rep.dim } A = \inf\{\text{gl.dim } (\mathrm{End}_A(M)) \colon M \text{ generator-cogenerator}\}$$

Auslander generator for A is a generator-cogenerator A-module M such that $\text{gl.dim}(\mathrm{End}_A(M) = \text{rep.dim } A)$.

Auslander's idea

The representation dimension of an algebra A would measure how far A is to be representation-finite.

Theorem (Auslander)

$$\text{rep.dim } A \leq 2 \iff A \text{ is representation-finite}$$

The proof of the above result inspired a criterium for the calculation of the representation dimension of an algebra. Made explicitly by Xi, C-Platzeck, Erdmann-Holm-Iyama-Schröer.

Auslander lemma.

Let M be a generator-cogenerator for A and $d \geq 0$. Then rep.dim $A \leq d+2$ if and only if for each A-module X, there exists an exact sequence

$$0 \longrightarrow M_d \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

with $M_i \in \text{add}M$ such that the sequence

$$0 \longrightarrow (M, M_d) \longrightarrow \cdots \longrightarrow (M, M_0) \longrightarrow (M, X) \longrightarrow 0$$

remains exact.

Representation dimension at most 3

Corollary

Let A be a representation-infinite algebra. Then rep.dim$A = 3$ if and only if there exists a generator-cogenerator M such that for each A-module X, there exists a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

with $M_i \in \operatorname{add} M$ such that

$$0 \longrightarrow (M, M_1) \longrightarrow (M, M_0) \longrightarrow (M, X) \longrightarrow 0$$

remains exact.

A strategy (to show rep.dim equals 3)

To find a generator-cogenerator M in such a way that a minimal right add$M$-approximation of each $X \in \operatorname{mod} A$ has kernel lying in add$M$.

Facts on Representation dimension

- (Iyama): rep.dim $A < \infty$
  (given $X \in \mathrm{mod}A$, Iyama constructs explicitly a module $Y \in \mathrm{mod}A$ such that $\mathrm{End}_A(X \oplus Y)$ is quasi-hereditary, and so of finite global dimension)
- (Igusa-Todorov): rep.dim $A \leq 3 \implies$ fin.dim $A < \infty$.
- For each $n \geq 2$, there are examples of algebras with rep.dim equal to n.

Torsionless-finite algebras

Theorem [Ringel]

Let A be a torsionless-finite algebra (that is, there are only finitely many, up to isomorphism, indecomposable modules which are submodules of projective modules). Then $\text{rep.dim} A \leq 3$.

An Auslander generator

$$M = (\oplus \text{torsionless modules}) \oplus (\oplus \text{quotients of injectives})$$

Torsionless-finite algebras

Special cases

- (Auslander):
  - Hereditary.
  - Algebras with $J^2 = 0$.
  - Algebras with $J^n = 0$ and A/J^{n-1} representation-finite.
- (C-Platzeck): Glued algebras.
  (right glued algebra: $\mathrm{Supp}(\mathrm{Hom}_A(-, A)$ is finite)
  (left glued algebra: $\mathrm{Supp}(\mathrm{Hom}_A(\mathrm{D}A, -)$ is finite)
- (Erdmann-Holm-Iyama-Schröer): Special biserial algebras.

Classes of algebras with rep.dim at most 3

- (Assem-Platzeck-Trepode): Tilted algebras.

$$M = A \oplus \; \mathrm{D}A \oplus \text{complete slice}$$

- (Oppermann) Quasitilted algebras (that is, gl.dim$A \leq 2$ and for each indecomposable module X, pd$X \leq 1$ or id$X \leq 1$)

- (C-Platzeck): Trivial extensions of hereditary algebras.

- (Gonzalez Chaio-Trepode) Cluster-concealed algebras.

The sides of a module category.

$$\mathcal{L}_A = \{X \in \ \mathrm{ind}A : \ \mathrm{pd}\, Y \leq 1 \ \ \forall Y \text{ predecessor of } X\}$$

$$\mathcal{R}_A = \{X \in \ \mathrm{ind}A : \ \mathrm{id}\, Y \leq 1 \ \ \forall Y \text{ successor of } X\}$$

Class of algebras with rep.dim at most 3

- (Assem-Platzeck-Trepode): Strict laura algebras. (that is, algebras with $\mathcal{L}_A \cup \mathcal{R}_A$ cofinite but not quasi-tilted).

$$M = A \oplus \ \mathrm{D}A \oplus \text{Ext-injectives in } \mathcal{L}_A \oplus$$

$$\oplus \text{Ext-projectives in } \mathcal{R}_A \oplus \text{indecomposable not in } \mathcal{L}_A \cup \mathcal{R}_A$$

Question

Do tame algebras have representation dimension at most three ?

Positive answers

- (Erdmann-Holm-Iyama-Schröer): Special biserial algebras.
- (Bocian-Holm-Skowroński): Domestic self-injective algebras socle equivalent to a weakly symmetric algebra of euclidean type.
- (Assem-Skowroński-Trepode): Self-injective algebras of euclidean type.

Left and right support algebras

Assem-Coelho-Wagner

The idea is to relate the representation dimension of an algebra A with those given by left and the right support algebras defined from some special subcategories. As a consequence, get results for laura, ada and Nakayama oriented pullbacks.

Strategy

The strategy is to split the module category into pieces and calculate the representation dimension of the algebras associated to them.

Some definitions

Definition

A trisection in $\text{ind}A$ is a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of disjoint subcategories such that

- $\text{ind}A = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.
- $\text{Hom}_A(\mathcal{C}, \mathcal{B}) = \text{Hom}_A(\mathcal{C}, \mathcal{A}) = \text{Hom}_A(\mathcal{B}, \mathcal{A}) = 0$.

Definition

We say that a subcategory $\mathcal{C}$ of $\text{ind}A$ is covariantly finite provided for each $X \in \text{ind}A$ there exists a morphism $f_X : X \longrightarrow Y$ with Y in $\mathcal{C}$ such that any morphism from X to a module in $\mathcal{C}$ factors through f_X (such f_X is called right $\mathcal{C}$-approximation). Dually for contravariantly finite.

Left and right support algebras

Given a subcategory $\mathcal{C}$ of $\mathrm{ind}A$, the (left) support algebra ${}_{\mathcal{C}}A$ is the endomorphism algebra of the direct sum of the isoclasses of the indecomposable projective modules lying in $\mathcal{C}$. Dually, we define the (right) support algebra $A_{\mathcal{C}}$.

Theorem 1 (Assem-Coelho-Wagner)

Let A be representation-infinite and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a trisection in $\mathrm{ind}A$. Assume $\mathcal{B}$ finite.

(a) If $\mathcal{C} \subseteq \mathcal{R}_A$ and $\mathrm{add}\mathcal{C}$ is covariantly finite, then

$$\mathrm{rep.dim}A = \max\{3, \mathrm{rep.dim}_{\mathcal{A}}A\}.$$

(b) If $\mathcal{A} \subseteq \mathcal{L}_A$ and $\mathrm{add}\mathcal{A}$ is contravariantly finite, then

$$\mathrm{rep.dim}A = \max\{3, \mathrm{rep.dim}A_{\mathcal{C}}\}.$$

New proof for Laura algebras

If A is a laura algebra, then rep.dim$A \leq 3$.

Proposition

Let A be representation-infinite algebra. If $A \in \mathrm{add}(\mathcal{L}_A \cup \mathcal{R}_A)$, then rep.dim$A = 3$.

ada algebras

If A is an ada algebra (that is, $A \oplus \mathrm{D}A \in \mathrm{add}(\mathcal{L}_A \cup \mathcal{R}_A)$), then rep.dim$A \leq 3$.

Theorem 2 (Assem-Coelho-Wagner)

Let A be representation-infinite and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a trisection in $\mathrm{ind}A$. If

(a) $(\mathcal{A}\cup\ \mathrm{ind}A_{\mathcal{C}})^c$ is finite and $\mathrm{ind}A_{\mathcal{C}}$ is closed under successors; or

(b) $(\mathrm{ind}_{\mathcal{A}A} \cup \mathcal{C})^c$ is finite and $\mathrm{ind}_{\mathcal{A}}A$ is closed under predecessors.

Then,

$$\mathrm{rep.dim}A \leq \max\{\mathrm{rep.dim}_{\mathcal{A}}A, \mathrm{rep.dim}A_{\mathcal{C}}\}.$$

Nakayama oriented pullback

Let R be the Nakayama oriented pullback of $A \longrightarrow B$ and $C \longrightarrow B$. Then

$$\text{rep.dim} R \leq \max\{\text{rep.dim} A, \text{rep.dim}\, C\}.$$

▸ References

Definition

An algebra A is iterated tilted if there exists a sequence of algebras $A_0 = H, A_1, \cdots, A_r = A$, where H is hereditary, and tilting A_i-modules T_i $(i = 1, \cdots, r-1)$ such that for each $j = 1, \cdots, r$, $A_j = \mathrm{End}(T_{j-1})$.

Theorem (C-Happel-Unger)

If A is an iterated tilted algebra, then rep.dim $A \leq 3$.

References

A description (Happel-Rickard-Schofield and Rickard)

$$A \text{ is iterated tilted} \overset{HRS}{\Longleftrightarrow} \mathrm{D}^b(A) \simeq_\Delta \mathrm{D}^b(H) \;\; H \text{ hereditary}$$

$$\overset{R}{\Longleftrightarrow} \exists \text{ tilting complex } T^\bullet \in \mathrm{D}^b(H) \text{ such that} A \simeq \mathrm{End}_{\mathrm{D}^b(H)} T^\bullet$$

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Obrigado!