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# Hahn's generalised problem and corresponding Appell polynomial sequences

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# Abstract

This thesis is devoted to some aspects of the theory of orthogonal polynomials, paying a special attention to the classical ones (Hermite, Laguerre, Bessel and Jacobi). The elements of a classical sequence are eigenfunctions of a second order linear differential operator with polynomial coefficients $\mathcal{L}$ known as the Bochner's operator. In an algebraic manner, a classical sequence is also caracterised through the so-called Hahn's property, which states that an orthogonal polynomial sequence is classical if and only if the sequence of its (normalised) derivatives is also orthogonal.

In the present work we show that an orthogonal polynomial sequence is classical if and only if any of its polynomials fulfils a certain differential equation of order $2k$, for some positive integer $k$. We thoroughly reveal the structure of such differential equation and, for each classical family, we explicitly present the corresponding $2k$-order differential operator $\mathcal{L}_k$. When we consider $k = 1$, we recover the Bochner's differential operator: $\mathcal{L}_1 = \mathcal{L}$. On the other hand, as a consequence of Bochner's result, any element of a classical sequence must be an eigenfunction of a polynomial with constant coefficients in powers of $\mathcal{L}$. As a result of the introduction of the so-called $A$-modified Stirling numbers (where $A$ indicates a complex parameter), we are able to establish inverse relations between the powers of the Bochner operator $\mathcal{L}$ and $\mathcal{L}_k$.

Afterwards, we proceed to the quadratic decomposition of an Appell sequence. The four polynomial sequences obtained by this approach are also Appell sequences but with respect to another lowering differential operator, denoted $\mathcal{F}_\varepsilon$, where $\varepsilon$ is either 1 or -1. Thus, we introduce and develop the concept of Appell sequences with respect to the operator $\mathcal{F}_\varepsilon$ (where, more generally, $\varepsilon$ denotes a complex parameter): the $\mathcal{F}_\varepsilon$-Appell sequences. Subsequently, we seek to find all the orthogonal polynomial sequences that are also $\mathcal{F}_\varepsilon$-Appell, which are, indeed,

the $\mathcal{F}_\varepsilon$-Appell sequences that satisfy Hahn's property respect to $\mathcal{F}_\varepsilon$. This latter consists of the Laguerre sequences of parameter $\varepsilon/2$, up to a linear change of variable. Inspired by this problem, we characterise all the $\mathcal{F}_\varepsilon$-classical sequences. While ferreting out the all $\mathcal{F}_\varepsilon$-classical sequences, apart from the Laguerre sequence, we find certain Jacobi sequences (with two parameters). The quadratic decomposition of Appell sequences with respect to other lowering operators is also considered and the results obtained are akin to the aforementioned ones attained in the analogous problem.

# Resumo


Esta tese versa sobre alguns aspectos da teoria dos polinómios ortogonais, com especial destaque para os polinómios clássicos (Hermite, Laguerre, Bessel and Jacobi). Os elementos de uma sucessão de polinómios clássica são funções próprias de um operador diferencial linear de segunda ordem de coeficientes polinomiais $\mathcal{L}$, conhecido como operador de Bochner. Algebricamente, uma sucessão de polinómios ortogonais toma a designação de clássica se a sucessão das suas derivadas (normalisadas) fôr também ela ortogonal: propriedade de Hahn.

Na presente dissertação, mostramos que uma sucessão de polinómios ortogonais pode ainda ser caracterizada através de uma equação diferencial de ordem par. A estrutura desta equação é cuidadosamente revelada, o que nos permite obter explicitamente uma expressão para o operador correspondente, digamos $\mathcal{L}_k$ (onde $k$ representa um inteiro positivo), para cada uma das famílias de polinómios clássicos. O operador de Bochner surge assim como um caso particular: $\mathcal{L}_1 = \mathcal{L}$. Por outro lado, e como consequência natural do resultado de Bochner, os elementos de uma sucessão clássica são igualmente funções próprias de um dado polinómio de coeficientes constantes nas potências de $\mathcal{L}$. Perante a introdução daqueles a que designamos como números de Stirling $A$-modificados (onde $A$ representa um parâmetro de valor complexo), torna-se possível estabelecer relações inversas entre as potências de $\mathcal{L}$ e os operadores previamente obtidos $\mathcal{L}_k$, mais concretamente, tal corresponde a afirmar que se passa a poder descrever uma qualquer potência de $\mathcal{L}$ através de uma soma finita nos operadores $\mathcal{L}_k$ para $k = 0, 1, 2, \ldots$, e, reciprocamente, $\mathcal{L}_k$ admite uma expansão finita em potências de $\mathcal{L}$.

Além disso, procedemos à decomposição quadrática de uma sucessão de Appell. As quatro sucessões assim obtidas são também elas munidas do caracter de Appell mas relativamente a

um novo operador diferencial que baixa em uma unidade o grau de um polinómio, denotado por $\mathcal{F}_\varepsilon$, onde $\varepsilon$ toma os valores 1 ou -1. Por conseguinte, introduzimos e desenvolvemos o conceito de sucessões de Appell relativamente a um tal operador $\mathcal{F}_\varepsilon$ (onde, mais geralmente, $\varepsilon$ representa um parâmetro de valor complexo), as quais tomam a designação de sucessões $\mathcal{F}_\varepsilon$-Appell. A questão da determinação de todas as sucessões $\mathcal{F}_\varepsilon$-Appell que são simultaneamente ortogonais surge com toda a legitimidade. Por outras palavras, procuramos todas as sucessões $\mathcal{F}_\varepsilon$-Appell que satisfazem a propriedade de Hahn, desta feita, relativamente ao operador $\mathcal{F}_\varepsilon$. Como solução para este problema, há a relatar, a menos de uma transformação linear, a sucessão de Laguerre de parâmetro $\varepsilon/2$. Induzidos por este problema, abordamos a caracterização das sucessões $\mathcal{F}_\varepsilon$-clássicas. Além das sucessões de Laguerre, encontramos ainda algumas sucessões de Jacobi. A decomposição quadrática de sucessões de Appell relativamente a outros operadores com a mesma natureza da derivada ou de $\mathcal{F}_\varepsilon$ é igualmente considerada, sendo de sublinhar que as conclusões daqui obtidas assemelham-se às obtidas no problema análogo anteriormente mencionado.

# Résumé

Cette thèse est consacrée à quelques aspects de la théorie des polynômes orthogonaux, avec une attention spéciale consacrée aux polynômes classiques (Hermite, Laguerre, Bessel et Jacobi). Les éléments d'une suite classique sont toujours fonctions propres d'un opérateur différentiel $\mathcal{L}$ du deuxième ordre à coefficients polynomiaux, connu comme l'opérateur de Bochner. Algébriquement, une suite orthogonale est qualifiée de classique si la suite des dérivées (normalisées) est aussi orthogonale: c'est la propriété de Hahn.

Dans ce mémoire, on montre qu'une suite classique peut être caractérisée à l'aide d'une équation différentielle d'ordre pair. La structure de l'équation différentielle est soigneusement décrite et, pour chaque famille classique, nous donnons explicitement l'expression de l'opérateur différentiel correspondant $\mathcal{L}_k$. Quand on considère $k = 1$ on retrouve l'opérateur de Bochner: $\mathcal{L}_1 = \mathcal{L}$. D'ailleurs, une conséquence du résultat de Bochner consiste à dire que chaque élément d'une suite classique est fonction propre d'un certain polynôme à coefficients constants de puissances de $\mathcal{L}$. Avec l'introduction de ce qu'on a appelé les nombres de Stirling $A$-modifiés (où $A$ représente un paramètre complexe), on décrit les puissances de $\mathcal{L}$ comme une somme fini de $\mathcal{L}_k$ pour $k = 0, 1, 2, \ldots$, et réciproquement.

Ensuite, on procède à la décomposition quadratique d'une suite d'Appell. Les quatre suites ainsi obtenues possèdent, elles aussi, le caractère d'Appell mais par rapport à un nouvel opérateur différentiel qui abaisse d'une unité le degré d'un polynôme, noté $\mathcal{F}_\varepsilon$, avec $\varepsilon$ égal soit à 1 ou à -1. Ainsi, on introduit et développe le concept des suites d'Appell par rapport à cet opérateur $\mathcal{F}_\varepsilon$ (où, plus généralement, $\varepsilon$ représente un paramètre à valeur complexes): les suites $\mathcal{F}_\varepsilon$-Appell. De façon naturelle, on cherche toutes les suites orthogonales qui sont aussi $\mathcal{F}_\varepsilon$-Appell, autrement dit, on veut déterminer toutes les suites $\mathcal{F}_\varepsilon$-Appell qui possèdent la

propriété de Hahn par rapport à l'opérateur $\mathcal{F}_\varepsilon$. La solution de ce problème consiste en les suites de Laguerre de paramètre $\varepsilon/2$, à une transformation affine près. Motivé par ce problème on caractérise toutes les suites $\mathcal{F}_\varepsilon$-classiques. A part les suites de Laguerre,on trouve certaines suites de Jacobi. La décomposition quadratique des suites d'Appell par rapports à d'autres opérateurs de la même nature que la dérivée ou $\mathcal{F}_\varepsilon$ est également considérée et les résultats obtenus ressemblent à ceux obtenus dans le problème analogue mentionné ci-dessus.

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# Contents

# Prelude

The coverage of this work concerns the theory of orthogonal polynomials and it includes the classical topics concerning this subject. With the important applications to probability and statistics, partition theory, combinatorics, sphere packing, stochastic processes, X-ray tomography, quantum scattering theory and nuclear physics, the subject flourished during the past century and is far from being finished, while its origins may be traced back to the Legendre's work on planetary motion. Despite the number of applications that may be found in the literature, this work deals with some theoretical aspects of the subject by adding to it a humble contribution.

A sequence of orthogonal polynomials (OPS) may be characterised in several ways, but the most typical feature is the second-order recurrence relation that any of its elements satisfy

$$
\begin{array}{l}
P_0(x) = 1 \quad ; \quad P_1(x) = x - \beta_0 \\
P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x) \ , \ n = 0, 1, 2, 3, \ldots \ ,
\end{array}
$$

where the sequence of numbers $\{\beta_n, \gamma_{n+1}\}_{n \geqslant 0}$ is commonly known as the sequence recurrence coefficients and necessarily $\gamma_{n+1} \neq 0$ for any integer $n \geqslant 0$. The classical topics such as *Hermite, Laguerre, Bessel, Jacobi, Gegenbauer (or Ultraspherical), Tchebyshev, Legendre, Al-Sallam-Carlitz* polynomials are all examples of orthogonal polynomial sequences and will be in discussion during the text.

The orthogonal polynomials named as *Hermite, Laguerre* (with one parameter), *Bessel* (with one parameter) and *Jacobi* (with two parameters, including the special cases of *Gegenbauer, Tchebyshev and Legendre* polynomials) are collectively called as classical polynomials. As a matter of fact, they represent the four distinct equivalence classes of classical polynomials, as it is expounded at the beginning of chapter 2. According to the result of Salomon Bochner [18] in 1929, the classical polynomial sequences are the only OPS whose elements are solutions of the second order homogeneous differential equation

$$\mathcal{F}[y](x) := \Phi(x)\, y''(x) - \Psi(x)\, y'(x) = \lambda_n y(x),\ n = 0, 1, 2, 3, \ldots,$$

where $\Phi$ is a polynomial with degree at most $2$, $\Psi$ a one degree polynomial and $\{\lambda_n\}_{n\geqslant 0}$ is a sequence of nonzero numbers with exception to $\lambda_0 = 0$, are the classical polynomial sequences. For the obvious reasons, the above given differential equation is commonly called as *Bochner's differential equation* and the associated differential operator as the Bochner's operator.

In an algebraic manner, an OPS $\{P_n\}_{n\geqslant 0}$ is said to be classical if and only the sequence of its derivatives $\{P_n'\}_{n\geqslant 0}$ is also an OPS. This characterisation is due to Wolfgang Hahn [52] and earned the name of "Hahn's property". This property encloses a major significance because it leads to the study of other classes of polynomials.

Instead of basing our study on hypergeometric properties of classical polynomials as many authors do, we have rather founded our exposures in a purely algebraic point of view, which has the merit of connecting the various characterisations in a natural way. Upon such point of view, the integral representation of a form is relegated to a secondary plan. The theoretical background in focus here is the theory of linear functionals (here systematically called as *forms*) deeply developed by Pascal Maroni [79, 85, 89] and intimately connected to the work of "umbral calculus" presented by Roman and Rota [97]. The main idea consists in privileging the intrinsic relations which can exist between the forms considered, by redirecting the problem to the dual space of the polynomial functions. In chapter 1 we give all the necessary background concerning the general theory of orthogonal polynomials to the understanding of the sequel. First, in section 1.1 the definitions and the operations obtained by transposition, that we naturally perform in the dual space, are given and in the following section some of the most important properties of such operations can be found. Section 1.3 is devoted to the introduction of the dual sequence; in section 1.4 the definition of regular orthogonality, as well as some of its most significant properties, is provided.

Based on the previous considerations, in chapter 2 we review different characterisations of classical polynomial sequences: the aforementioned Bochner and Hahn's properties, the Rodrigues type formula and, among others, structural relations. These are presented in section 2.1. In theorem 2.2.1, we show, for any integer $k \geqslant 1$, that the elements of any classical sequence must be solution of a $2k$-order differential equation of the type

$$\mathcal{F}_k[y](x) := \sum_{\nu=k}^{2k} l_\nu(x)\, \frac{d^\nu}{dx^\nu} y(x) = \Xi_n(k) y(x)$$

where the structure of the polynomial coefficients $l_\nu(\cdot)$ (with $k \leqslant \nu \leqslant 2k$), as well as the expressions of the eigenvalues sequence $\{\Xi_n(k)\}_{n\geqslant 0}$ is thoroughly revealed, permitting to obtain explicit expressions in terms of the polynomial coefficients of the Bochner operator $\mathcal{F}$. Concerning the reciprocal condition, we demonstrate in theorem 2.4.1, that if the elements of an OPS eigenfunctions of an operator alike $\mathcal{F}_k$, then it must be a classical sequence, achieving one of the goals of this thesis: the generalisation of the Bochner condition about de classical polynomials. Subsequently, we expound how the even order differential operator $\mathcal{F}_k$ may be written as a polynomial with constant coefficients in the powers of the operator $\mathcal{F}$ and, conversely, we also express any power of $\mathcal{F}$ in terms of a sum of the operators $\mathcal{F}_k$. In brief, we establish inverse relations between powers of $\mathcal{F}$ and its "factorials" $\mathcal{F}_k$. The bridge between these two operators is attained through the Stirling numbers. In section 2.2.3 we review this concept, which is sufficient to accomplish the study of the cases of Hermite and Laguerre classical sequences, whereas the cases of Bessel or Jacobi require the introduction and development of the concept of the so-called *$A$-modified Stirling numbers* (where $A$ represents any complex parameter). The analysis is guided separately in section 2.2.4 for each classical family. We end this chapter with a generalisation of Rodrigues-type formula.

Chapter 3 gives a prominent emphasis to Appell sequences. Since the work of Angelescu [7], it is well known that the sole classical polynomial sequence being also Appell is essentially the Hermite sequence (that is, up to a linear change of variable). The even and odd terms of this symmetric sequence may be expressed through the Laguerre polynomials of parameter $-1/2$ and $1/2$, respectively. Inspired by this fact, in section 3.1 we describe the even and odd terms of an Appell sequence (not necessarily orthogonal or symmetric) by means of four other polynomial sequences. Such procedure is known as the quadratic decomposition of a polynomial sequence and it frequently brings to light important information about the original sequence. The four sequences obtained by this approach are also Appell sequences but with respect to another lowering operator $\mathcal{F}_\varepsilon$ where $\varepsilon$ is either 1 or -1. The concept of Appell sequences may be broaden to Appell sequences with respect to other lowering operators $\mathcal{O}$: the $\mathcal{O}$-Appell sequences. Thus, we characterise the $\mathcal{F}_\varepsilon$-Appell sequences (where $\varepsilon$ represents a parameter belonging to the field of complex numbers) and while, in section 3.3, ferreting out all those ones being also orthogonal, we solely find the Laguerre polynomials with parameter $\varepsilon/2$, up to a linear change of variable. The quadratic decomposition of Appell sequences with respect to other lowering operators such as $\mathcal{F}_\varepsilon$ or the $q$-derivative is considered as well, and the results obtained are akin to the ones attained in the analogous aforementioned problem.

The orthogonal $\mathcal{F}_\varepsilon$-Appell sequences are part of a collection of orthogonal sequences satisfying the Hahn's property but with respect to the operator $\mathcal{F}_\varepsilon$: the $\mathcal{F}_\varepsilon$-classical sequences. This brings us to chapter 4, where the main purpose is to characterise these type of sequences according to the framework carried out in the study of $D$-classical sequences or of other classical sequences with respect to the Hahn's operators. In the course of seeking out all the $\mathcal{F}_\varepsilon$-classical sequences, we find the Laguerre and Jacobi polynomials, up to a linear change of variable. The inherent technical difficulties were overcome by slightly modifying the techniques already used. In this process, the symbolic computational language *Mathematica*© was a reliable instrument. We believe that the reasoning behind the resolution of this problem may be adapted to solve other Hahn's problems.

The numbering system used in this work is the common one whereby (2.3.6) refers to the 6th numbered equation in section 3 of chapter 2. An analogous scheme is followed for theorems, propositions, lemmas and corollaries, but not for definitions or remarks. The practice of Halmos of indicating the end of a proof by the symbol □ is adopted. All the references in the text are in the bibliography chapter, ordered alphabetically, wherein [10] refers to the 10th entry in that chapter; the bibliography has no pretensions of completeness.

In case of misprints, any errors or inadequacies that remain, I assume full responsibility. I also hope that the lecture of this thesis is both pleasant and enjoyable, despite the fact that some parts of the thesis may not be of straightforward reading for non-specialists.

# CHAPTER 1

## Background and general features

As usual, we use the symbol $\mathbb{N}$ to represent the set of all nonnegative integers, $\mathbb{R}$ for the set of all real numbers and $\mathbb{C}$ for the set of all complex numbers. The set $\mathbb{N}$ without $0$ will be denoted by $\mathbb{N}^*$, and similarly, $\mathbb{R}^*$ and $\mathbb{C}^*$ represent the set $\mathbb{R}$ and $\mathbb{C}$ with $0$ excluded, respectively. Throughout the text we often use the symbol $n$ to represent an integer and, for instance, we will simply write $n \geqslant c$ which means the integers $n$ bigger or equal to $c$, unless the context requires more precision. The derivative of a function $f$ will be denoted either as $Df$ or as $f'$ and by $D^k f$ or $f^{(k)}$ we mean the $k$-th order derivative of $f$, recursively defined as $f^{(k)} = \left(f^{(k-1)}\right)'$ for any $k \in \mathbb{N}^*$.

The vector space of polynomials with coefficients in $\mathbb{C}$ will be denoted by $\mathcal{P}$. Consider $\mathcal{P}_n$, with $n \in \mathbb{N}$, to be the subspace of $\mathcal{P}$ of polynomials with degree lower than or equal to $n$. Naturally, $\mathcal{P}_n$ is the vector space spanned by the set $\{x^k\}_{0 \leqslant k \leqslant n}$, so an element $f$ of $\mathcal{P}_n$ may be expressed like $f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$, with $a_\nu \in \mathbb{C}$, for all the integers $\nu$ not exceeding $n$. In a finite dimensional space all the norms are equivalent, therefore, without loss of generality, we may define the norm $\|f\|_n := \sum_{\tau=0}^{n} |a_\tau|$ for an arbitrary polynomial $f \in \mathcal{P}_n$, such that $f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$, $n \in \mathbb{N}$. Since a finite dimensional normed space is always complete, $\mathcal{P}_n$ equipped with the norm $\|\cdot\|_n$, with $n \in \mathbb{N}$, is a Banach space (hence, a Frechet space). $\mathcal{P}$ may be viewed as the union of an increasing sequence of the subspaces $\mathcal{P}_n$, *i.e.* $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$. Each $\mathcal{P}_n$ is isomorphically embedded in $\mathcal{P}_{n+1}$, which means that the topology induced by $\mathcal{P}_{n+1}$ on $\mathcal{P}_n$ is identical to the topology initially given on $\mathcal{P}_n$. In addition $\mathcal{P}_n$ is closed in $\mathcal{P}_{n+1}$. So $\mathcal{P}$ is equipped with the topology of strict inductive limit of the Frechet subspaces $\mathcal{P}_n$. In the book of Trèves [104] a detailed survey about these concepts may be found, but an interesting reading may be followed in the first volume of the book of Khoan [60].

Consider $\mathcal{P}^*$ to be the algebraic dual of $\mathcal{P}$, that is, the set of all *linear functionals* or *forms* $u : \mathcal{P} \to \mathbb{C}$. We will denote by $\langle u, f\rangle$ the effect of a form $u \in \mathcal{P}^*$ on a polynomial $f \in \mathcal{P}$. The topological dual of $\mathcal{P}$, represented by $\mathcal{P}'$, consisting of all the continuous linear functionals $u : \mathcal{P} \to \mathbb{C}$, is a vector subspace of $\mathcal{P}^*$. The weak topology of $\mathcal{P}'$ is defined by the system of seminorms

$$|u|_n := \sup_{\nu \leqslant n} |\langle u, x^\nu\rangle|$$

and it equals the strong dual topology [104, pp 195-201]. Moreover, $\mathcal{P}'$ is a *Frechet* space and it equals $\mathcal{P}^*$. Throughout the text we refer to $\mathcal{P}'$ as the dual space of $\mathcal{P}$, whose elements we will systematically call as *forms* instead of *linear functionals*. The effect of a form $u$ on the polynomial $x^n$ is represented as $(u)_n := \langle u, x^n\rangle$, $n \in \mathbb{N}$, and it is called the moment of $u$ of order $n$. Indeed, any form $u$ may be described by its moment sequence $\{(u)_n\}_{n\in\mathbb{N}}$.

## 1.1 Some elementary operations in the dual

Some of the most common linear operations in $\mathcal{P}$ and the theory of orthogonal polynomials are inextricable. The theory developed in this text is essentially based on operations in forms, which are induced by the existent operations in the space $\mathcal{P}$. Precisely, a linear operator $T : \mathcal{P} \to \mathcal{P}$ (that maps elements of $\mathcal{P}$ into itself) has a *transpose* ${}^tT : \mathcal{P}' \to \mathcal{P}'$ defined by

$$\langle\ {}^tT(u), f\ \rangle := \langle\ u, T(f)\ \rangle, \quad u \in \mathcal{P}', f \in \mathcal{P}, \tag{1.1.1}$$

and ${}^tT$ is a linear application[1]. By transposition of the usual operations defined on $\mathcal{P}$, we are able to define the following linear operations in $\mathcal{P}'$:

**Left-multiplication of a form $u$ by a polynomial $f$**, denoted as $fu$, is given by

$$\langle fu, p\rangle := \langle u, fp\rangle, \quad p \in \mathcal{P}, \tag{1.1.2}$$

where $p \mapsto fp$ is from $\mathcal{P}$ into $\mathcal{P}$. In particular,

$$\big(fu\big)_n = \sum_{\nu=0}^{m} a_\nu (u)_{\nu+n} \quad \text{with} \quad f(x) = \sum_{\nu=0}^{m} a_\nu x^\nu,\ n, m \in \mathbb{N}. \tag{1.1.3}$$

[1]For a more detailed discussion but rather simple, see Khoan [60], pp 72-74

**Derivative of a form** $u$, which we denote by $u' := Du$ is defined as

$$\langle u', p\rangle := -\langle u, p'\rangle, \quad p \in \mathcal{P}, \tag{1.1.4}$$

Thus, the differentiation operator on forms $D$ is minus the transpose of the differentiation operator $D$ on polynomials. In particular, we have

$$\big(u'\big)_n = -n(u)_{n-1}, \quad n \in \mathbb{N} \text{ and } u_{-1} = 0.$$

Derivatives of higher order of a given form $u$, are recursively defined as follows:

$$\langle u^{(k)}, p\rangle := -\langle u^{(k-1)}, p'\rangle, \quad p \in \mathcal{P},\ k \in \mathbb{N}^*.$$

Therefore we have

$$\langle u^{(k)}, p\rangle = (-1)^k \langle u, p^{(k)}\rangle, \quad p \in \mathcal{P},\ k \in \mathbb{N}^*.$$

In particular, the moments of $u^{(k)}$ are given by

$$\Big(u^{(k)}\Big)_n = (-1)^k \prod_{\nu=0}^{k-1} (n-\nu)\ (u)_{n-k}, \quad n \in \mathbb{N},\ k \geqslant, \text{ with } (u)_{-\mu} = 0, \mu \geqslant 1.$$

**Translation of a form** $u$ by $b \in \mathbb{C}$, is denoted as $\tau_b u$ and is given by

$$\langle \tau_b u, p\rangle := \langle u, \tau_{-b}\, p\rangle, \quad p \in \mathcal{P}, \tag{1.1.5}$$

where $\tau_{-b}$ is a linear map of $\mathcal{P}$ into itself defined by $p(x) \mapsto \big(\tau_{-b}\, p\big)(x) = p(x+b)$. In particular, we have

$$(\tau_b\, u)_n = \sum_{\nu+\mu=n} \frac{n!}{\nu!\,\mu!}\, (u)_\nu\, b^\mu, \quad n \in \mathbb{N}.$$

**Homotety of a form** $u$ by $a \in \mathbb{C}^*$, denoted as $h_a u$, is defined by

$$\langle h_a u, p\rangle := \langle u, h_a p\rangle, \quad p \in \mathcal{P}, \tag{1.1.6}$$

where $h_a$ is a linear map of $\mathcal{P}$ into itself defined by $p(x) \mapsto \big(h_a p\big)(x) = p(a\,x)$. The moments of the form $h_a u$ are

$$(h_a u)_n = a^n\ (u)_n, \quad n \in \mathbb{N}.$$

**Division of a form** $u$ **by a first degree polynomial** : $(x-c)^{-1}\, u, \quad c \in \mathbb{C}$

$$\langle (x-c)^{-1}\, u, p\rangle := \langle u, \vartheta_c p\rangle, \quad p \in \mathcal{P}, \tag{1.1.7}$$

where $\vartheta_c$ is a linear map of $\mathcal{P}$ into itself defined by

$$p(x) \mapsto \big(\vartheta_c p\big)(x) := \frac{p(x)-p(c)}{x-c} \quad . \tag{1.1.8}$$

The division of a form by a polynomial $R$ of higher degree is recursively defined through

$$\Big((x-c)R(x)\Big)^{-1} u = (x-c)^{-1}\big(R^{-1}(x)\, u\big)$$

**Cauchy product of two forms** $uv,\ u,v \in \mathcal{P}'$

$$\langle uv, p\rangle := \langle u, vp\rangle \quad , \quad p \in \mathcal{P},$$

where

$$\big(vp\big)(x) := \langle v, \frac{xp(x)-\zeta p(\zeta)}{x-\zeta}\rangle$$

corresponds to the **right-multiplication** of a form by a polynomial. In particular, the moments of $uv$ are given by

$$\big(u\,v\big)_n = \sum_{\nu+\mu=n} (u)_\nu (v)_\mu \ , \quad n \in \mathbb{N}.$$

When $v$ is such that $uv = \delta$ (*Dirac delta*), where $\delta = \delta_0$, $\langle \delta_0, f\rangle = f(0)$, then $v$ is called the *inverse* of $u$, $v = u^{-1}$. The inverse exists if and only if $(u)_0 \neq 0$.

Any surjective linear application $T$ on $\mathcal{P}$ has a one-to-one (injective) transpose ${}^tT$. In particular, $Du = 0$ if and only if $u = 0$.

In the case where a linear application $T$ is an isomorphism of $\mathcal{P}$ into itself, its transpose ${}^tT$ is also an isomorphism of $\mathcal{P}'$ into itself and the reciprocal of ${}^tT$ corresponds to the transpose of $T^{-1}$. For instance any affine function $ax+b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, gives rise to an isomorphism $T = h_a \circ \tau_{-b} : \mathcal{P} \longrightarrow \mathcal{P}$ defined by $\big(Tp\big)(x) = p(ax+b)$, for $p \in \mathcal{P}$, and the inverse operator $T^{-1}$ is associated with the affine transformation $x/a - b/a$, so $T^{-1} = \tau_b \circ h_{a^{-1}}$. In this case we have ${}^tT = \tau_b \circ h_a$ and ${}^t\big(T^{-1}\big) = \big({}^tT\big)^{-1} = h_{a^{-1}} \circ \tau_{-b}$.

## 1.2 Some properties of operations in $\mathcal{P}$ and $\mathcal{P}'$

The properties listed below are by far well known and they may be found in the existent bibliography (Loureiro [74], Maroni [78, 79, 82, 84, 85], Roman and Rota [97])

For any $f \in \mathcal{P}$, $u \in \mathcal{P}'$, $a \in \mathbb{C}^*$ , $b \in \mathbb{C}$, we have:

$$(fu)' = f\,u' + f'\,u \;, \tag{1.2.1}$$
$$(\tau_b f)(\tau_b u) = \tau_b(fu) \;, \tag{1.2.2}$$
$$(h_{a^{-1}} f)(h_a u) = h_a(fu) \;, \tag{1.2.3}$$
$$\big(\tau_b u\big)' = \tau_b u' \;, \tag{1.2.4}$$
$$\big(h_a u\big)' = a^{-1} h_a u' \;. \tag{1.2.5}$$

Concerning the division of a form by a first degree polynomial combined with the derivative and the product, the following properties are valid for any $f \in \mathcal{P}$, $u \in \mathcal{P}'$, $b, c, d \in \mathbb{C}$:

$$\begin{aligned}
&\big(\vartheta_0 \tau_{-b} f\big)(x) = \big(\tau_{-b} \vartheta_b f\big)(x)\\
&\vartheta_c D = D\vartheta_c + \vartheta_c^2 \;,\\
&\vartheta_c\, \vartheta_d = \vartheta_d\, \vartheta_c = (d-c)^{-1}(\vartheta_d - \vartheta_c) \;,\; c \neq d,\\
&\big((x-c)^{-1}u\big)' = (x-c)^{-1}u' - (x-c)^{-2}u \;,\\
&f\big((x-c)^{-1}u\big) = (x-c)^{-1}(fu) + \langle u, \vartheta_c f\rangle \delta_c \;,\\
&(x-c)^{-1}\big(fu\big) = f(c)\big((x-c)^{-1}u\big) + \big(\vartheta_c f\big)u - \langle u, \vartheta_c f\rangle \delta_c
\end{aligned}$$

where $\delta_c = \tau_c\, \delta \in \mathcal{P}'$. As particular cases of the two last identities, but rather important to notice here, are

$$(x-c)\big((x-c)^{-1}u\big) = u \qquad ; \qquad (x-c)^{-1}\big((x-c)u\big) = u - (u)_0 \delta_c \;. \tag{1.2.6}$$

In consequence of the definition of the (left) product of a polynomial by a form and the transpose of the derivative operator, for any polynomial $f$ and any form $u$ the equality holds

$$D^k\big(f\,u\big) = \sum_{\nu=0}^{k} \binom{k}{\nu} \big(D^\nu f\big)\;\big(D^{k-\nu}u\big),\; k \in \mathbb{N}^* \;. \qquad (\textit{\underline{Leibniz derivation formula}})$$

## 1.3 Polynomial sequences and dual sequences

A discrete set of polynomials $B_n$ is called a polynomial set and denoted by $\{B_n\}_{n\in\mathbb{N}}$ when the degree of each of its elements is lower or equal to a nonnegative integer $n$. When the set $\{B_n\}_{n\in\mathbb{N}}$ spans $\mathcal{P}$, which occurs if $\deg B_n = n$, $n \in \mathbb{N}$, then it will be called a **polynomial sequence**, or, in short, PS . The elements of a PS $\{B_n\}_{n\in\mathbb{N}}$ can be taken monic (*i.e.* $B_n(x) = x^n + b_n$ with $\deg b_n < n$, for $n \geqslant 1$ and $B_0 = 1$) and, in this case, $\{B_n\}_{n\in\mathbb{N}}$ is said to be a **monic polynomial sequence**, hereafter abbreviated to MPS. The Euclidean division of the polynomial $B_{n+1}(x)$ by $B_n(x)$, with $n \in \mathbb{N}$, leads to a *structure relation of the MPS* $\{B_n\}_{n\in\mathbb{N}}$, more precisely there are two complex (number) sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\chi_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ such that

$$\begin{cases} B_0(x) = 1 \quad ; \quad B_1(x) = x - \beta_0 \\ B_{n+2}(x) = (x-\beta_{n+1})B_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu} B_\nu(x), \quad n \in \mathbb{N}. \end{cases} \tag{1.3.1}$$

It is always possible to associate to a MPS $\{B_n\}_{n\in\mathbb{N}}$ a unique sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in \mathcal{P}'$, $n \in \mathbb{N}$, which is called the **dual sequence** of $\{B_n\}_{n\in\mathbb{N}}$, and is defined by the biorthogonal condition

$$\langle u_n, B_m \rangle = \delta_{n,m}, \quad n, m \geqslant 0, \tag{1.3.2}$$

where $\delta_{n,m}$ represents the *Kronecker's symbol* (it equals $1$ when $n = m$ and $0$ otherwise), see Brezinski [20].

**Example.** The dual sequence associated to the MPS $\{x^n\}_{n\in\mathbb{N}}$ corresponds to the sequence $\{\frac{(-1)^n}{n!} D^n\delta\}_{n\in\mathbb{N}}$.

Based on the definition of dual sequence, the relation (1.3.1) provides

$$\beta_n = \langle u_n \, , \, x\, B_n \rangle \, , \quad \text{for } \; n \in \mathbb{N}, \tag{1.3.3}$$

$$\chi_{n,\nu} = \langle u_\nu \, , \, x\, B_{n+1} \rangle \, , \quad \text{for } \; n \in \mathbb{N}. \tag{1.3.4}$$

The dual sequence of a given MPS forms a basis $\mathcal{P}'$. Given an element of $\mathcal{P}'$, one might be interested in expressing it as a linear combination of elements of the dual sequence of a certain MPS. So, we recall a useful result.

**Lemma 1.3.1.** *Let $\{B_n\}_{n\in\mathbb{N}}$ be a MPS and $\{u_n\}_{n\in\mathbb{N}}$ the corresponding dual sequence. For any $u \in \mathcal{P}'$ and any integer $m \geqslant 1$, the following statements are equivalent.*

(a) $\langle u, B_{m-1} \rangle \neq 0$, $\langle u, B_n \rangle = 0$, $n \geqslant m$.

(b) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leqslant \nu \leqslant m-1$, $\lambda_{m-1} \neq 0$ *such that* $u = \displaystyle\sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$ .

*Furthermore,* $\lambda_\nu = \langle u, B_\nu \rangle$ , $0 \leqslant \nu \leqslant m-1$ *[85].*

Naturally, whenever for a given form $u$ and a given MPS $\{B_n\}_{n\in\mathbb{N}}$ we have

$$\langle u, B_n \rangle = 0 \, , \quad n \geqslant 0,$$

then necessarily $u = 0$.

The previous lemma is at the basis of a number of ensuing results. In particular, it is the key to derive the dual sequence of a MPS obtained from another MPS through elementary operations such as linear transformation or differentiation, among others. It is worthy to recall two examples already given by Maroni [78, 82, 85].

1. The sequence $\{\widetilde{B}_n\}_{n\in\mathbb{N}}$ defined by $\widetilde{B}_n(x) := a^{-n} B_n(ax+b)$ with $a \neq 0$ is a MPS and its corresponding dual sequence $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ is such that

$$\widetilde{u}_n = a^n \left(h_{a^{-1}} \circ \tau_{-b}\right) u_n \,, \quad n \in \mathbb{N}. \tag{1.3.5}$$

2. The normalised derivative sequence $\{B_n^{[1]}\}_{n\in\mathbb{N}}$, defined by

$$B_n^{[1]}(x) := \frac{1}{n+1}\, B'_{n+1}(x), \quad n \in \mathbb{N}, \tag{1.3.6}$$

is still a MPS and the corresponding dual sequence $\{u_n^{[1]}\}_{n\in\mathbb{N}}$ satisfies

$$\left(u_n^{[1]}\right)' = -(n+1)\, u_{n+1}, \quad n \in \mathbb{N}. \tag{1.3.7}$$

The sequence of higher order derivatives, $\{B_n^{[k]}\}_{n\in\mathbb{N}}$, with $k \geqslant 0$, is recursively defined

$$B_n^{[k+1]}(x) := \frac{1}{n+1}\, \left(B_{n+1}^{[k]}(x)\right)', \quad n \in \mathbb{N}, \tag{1.3.8}$$

and the corresponding dual sequence, denoted as $\{u_n^{[k]}\}_{n\in\mathbb{N}}$, with $k \geqslant 0$, fulfils

$$\left(u_n^{[k+1]}\right)' = -(n+1)\, u_{n+1}^{[k]}, \quad n \in \mathbb{N}. \tag{1.3.9}$$

By finite induction it is easy to deduce

$$\left(u_n^{[k]}\right)^{(k)} = (-1)^k \prod_{\mu=1}^{k} (n+\mu)\; u_{n+k}\,, \quad n \in \mathbb{N},\ k \in \mathbb{N}^*. \tag{1.3.10}$$

A thorough description of the sequences just presented may be found in Loureiro [74].

## 1.4 Regular orthogonality

We now turn to a formal discussion of the regular forms and the corresponding orthogonal polynomial sequences. The forthcoming definitions and results are crucial for the sequel, consequently requiring to be stated formally.

**Definition 1.4.1.** A PS $\{B_n\}_{n\in\mathbb{N}}$ is said to be an **orthogonal polynomial sequence** (OPS) with respect to a form $u$ provided for all integers $m, n \in \mathbb{N}$,

$$\langle u, B_n B_m\rangle = k_n \, \delta_{n,m} \quad \text{with} \quad k_n \neq 0. \tag{1.4.1}$$

In this case, $u$ is called a **regular** form.

It is well known that if $\{B_n\}_{n\in\mathbb{N}}$ is an OPS with respect to $u$, then so is $\{c_n\, B_n\}_{n\in\mathbb{N}}$, no matter the choice for the nonzero constants $c_n$, $n \in \mathbb{N}$. Conversely, an OPS $\{B_n\}_{n\in\mathbb{N}}$ with respect to a certain regular form may be uniquely determined if it satisfies an additional condition fixing the leading coefficient of each $B_n$. Therefore, given a regular form, we shall single out a particular OPS by specifying the value of the leading coefficient of each polynomial. In order to avoid further ambiguities, we will, as far as possible, require the OPS to be monic, which we refer to as **monic orthogonal polynomial sequence** or, in short, **MOPS**. Among the vast collection of works concerning the orthogonal polynomials; among them we quote: Chihara [26], Maroni [78, 82], Roman and Rota [97], Szegö [102].

Sometimes, unless there is danger of ambiguity, we loosely refer to an "MOPS $\{B_n\}_{n\in\mathbb{N}}$ with respect to a form $u$" as "$\{B_n\}_{n\in\mathbb{N}}$ orthogonal for $u$" or "$u$ a regular form of the MOPS $\{B_n\}_{n\in\mathbb{N}}$".

There is a large number of properties satisfied by all the MOPS. Among them we recall those that are undoubtedly fundamental to the forthcoming developments.

As a consequence of the definition of a regular form, $(u)_0 \neq 0$ and, in this case, $u$ is proportional to $u_0$, the first element of the dual sequence of $\{B_n\}_{n\in\mathbb{N}}$. Furthermore, the elements of the dual sequence of a MOPS $\{B_n\}_{n\in\mathbb{N}}$ are such that

$$u_n = \left(\langle u_0, {B_n}^2\rangle\right)^{-1} B_n u_0, \quad n \in \mathbb{N}, \tag{1.4.2}$$

and any three consecutive polynomials of $\{B_n\}_{n\in\mathbb{N}}$ are related through the following second-order recurrence relation

$$\begin{aligned} &B_0(x) = 1 \quad ; \quad B_1(x) = x - \beta_0 \,, \\ &B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1} B_n(x) \,, \quad n \in \mathbb{N}, \end{aligned} \tag{1.4.3}$$

where $\beta_n = \dfrac{\langle u_0, x\, {B_n}^2\rangle}{\langle u_0, {B_n}^2\rangle}$ and $\gamma_{n+1} = \dfrac{\langle u_0, {B_{n+1}}^2\rangle}{\langle u_0, {B_n}^2\rangle}$ for any nonnegative integer $n$. Sometimes during the text, we refer to the pair $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$ fulfilling (1.4.3) as **recurrence coeffcients** of the MOPS $\{B_n\}_{n\in\mathbb{N}}$. An outcome of this second order recurrence relation consists in the fact that any two consecutive elements of a MOPS cannot have roots in common.

Obviously, (1.4.3) is a particular case of the structural relation given in (1.3.1), and it is natural to conclude $\chi_{n,\nu} = \gamma_{n+1}\, \delta_{n,\nu}$, for $0 \leqslant \nu \leqslant n$ and $n \in \mathbb{N}$.

Another way for showing the orthogonality of a given MPS (particularly important in what concerns Chapter 4) is stated in the next result, whose may be found in [26, 74, 79]

**Proposition 1.4.2.** *[26, 74, 79] An MPS $\{B_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to the form $u$ if and only if there is a MPS $\{Q_n\}_{n\in\mathbb{N}}$ such that*

$$\langle u, Q_m B_n\rangle = 0\ , \quad \textit{for any } m \in \mathbb{N} \textit{ and } 0 \leqslant m \leqslant n-1\ ,$$
$$\langle u, Q_n B_n\rangle \neq 0\ , \quad \textit{for any } n \in \mathbb{N}.$$

Besides when we are operating with regular forms, an important property comes out.

**Lemma 1.4.3.** *[85] For any regular form $u$ and any polynomial $\phi$ such that $\phi\, u = 0$, necessarily $\phi = 0$.*

One might wonder when a form $u$ ought to be regular, or, in other words, when does a MPS orthogonal with respect to $u$ exist. Indeed, a form $u$ is regular if and only if the *Hankel determinant* of $u$, denoted as $\Delta_n(u) := \det\left[(u)_{\nu+\mu}\right]_{0\leqslant\nu,\mu\leqslant n}$, is different from zero. In this case, the elements of the associated orthogonal sequence $\{B_n\}_{n\in\mathbb{N}}$ admit the representation

$$B_n(x) = \frac{1}{\Delta_{n-1}(u_0)} \begin{vmatrix} 1 & (u_0)_1 & \cdots & (u_0)_{n-1} & (u_0)_n \\ (u_0)_1 & (u_0)_2 & \cdots & (u_0)_n & (u_0)_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (u_0)_{n-1} & (u_0)_n & \cdots & (u_0)_{2n-2} & (u_0)_{2n-1} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}\ , \quad n \in \mathbb{N},$$

with the convention $\Delta_{n-1}(u_0) = 1$, as used in chapter 2 in the book of C. Brezinski [21] and also in the book of T. Chihara [26]. Moreover, it is also possible to express $B_n(x) = x^n + b_n x^{n-1} + \ldots$, where the set of coefficients $\{b_n\}_{n\in\mathbb{N}}$ is such that $\beta_n = b_n - b_{n+1}$, for $n \in \mathbb{N}$. Therefore, it turns out

$$\beta_n = \frac{\Delta^*_{n+1}(u_0)}{\Delta_n(u_0)} - \frac{\Delta^*_n(u_0)}{\Delta_{n-1}(u_0)}\ , \quad n \in \mathbb{N},$$

where $\Delta^*_n(u_0)$ represents the $(n \times n)$-determinant obtained from $\Delta_n(u_0)$ by deleting its $(n+1)^{\text{th}}$ row and the $n^{\text{th}}$ column, under the convention $\Delta^*_0(u_0) = 0$. In addition, we have $\Delta_{n+1}(u_0) = \Delta_n(u_0)\langle u_0, B^2_{n+1}\rangle$ for any integer $n \in \mathbb{N}$, yielding

$$\gamma_{n+1} = \frac{\Delta_{n-1}(u_0)\Delta_{n+1}(u_0)}{\left(\,\Delta_n(u_0)\,\right)^2}\ , \quad n \in \mathbb{N}.$$

At last we recall the so-called *Christoffel-Darboux* formula fulfilled by any orthogonal sequence (Christoffel [29]) $\{B_n\}_{n\in\mathbb{N}}$:

$$\frac{B_{n+1}(x)B_n(y) - B_n(x)B_{n+1}(y)}{x-y} = \sum_{\nu=0}^{n} \frac{\langle u_0, B_n^2\rangle}{\langle u_0, B_\nu^2\rangle}\, B_\nu(x)\; B_\nu(y)\ ,\ n\in\mathbb{N},\ x,y\in\mathbb{C}.$$

For further reading, please consult Brezinski [19].

The second order recurrence relation (1.4.3) permits to deduce that $\{B_n\}_{n\in\mathbb{N}}$ is real if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} \in \mathbb{R}^*$, for any $n \in \mathbb{N}$. This is to say that all the moments of the regular form $u_0$ are real, *i.e.* $(u_0)_n \in \mathbb{R}$, $n \in \mathbb{N}$. A necessary and sufficient condition to $\{B_n\}_{n\in\mathbb{N}}$ (*resp.* the form $u_0$) be *positive definite* is given by the conditions $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$, for $n \in \mathbb{N}$, which corresponds to have $\Delta_{n+1}(u_0) > 0$, for $n \in \mathbb{N}$. In an equivalent way, we have $\langle u_0, p\rangle > 0$ for any $p \in \mathcal{P} - \{0\}$ such that $p(x) \geqslant 0$, $x \in \mathbb{R}$. Likewise, the sequence $\{B_n\}_{n\in\mathbb{N}}$ (*resp.* the form $u_0$) is called *negative definite* when $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} < 0$, for $n \in \mathbb{N}$. Equivalently, the form $u_0$ is negative definite if and only if it is real and $\Delta_{4n+1}(u_0) < 0$, $\Delta_{4n+2}(u_0) < 0$, $\Delta_{4n+3}(u_0) > 0$, $\Delta_{4n+4}(u_0) > 0$, $n \in \mathbb{N}$ (Chihara [26]).

**Example.** Any affine transformation $T = h_a \circ \tau_b$ $(a \in \mathbb{C}^*,\, b \in \mathbb{C})$ preserves the orthogonality of a polynomial sequence. More precisely, if $\{B_n\}_{n\in\mathbb{N}}$ represents a MOPS with respect to $u_0$, then so is the sequence $\{\widetilde{B}_n\}_{n\in\mathbb{N}}$ defined on page 27 and the corresponding regular form is $\widetilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})\, u_0$. Trivially, its recurrence coefficients, denoted as $(\widetilde{\beta}_n, \widetilde{\gamma}_{n+1})_{n\in\mathbb{N}}$, are given by

$$\widetilde{\beta}_n = \frac{\beta_n - b}{a} \quad ; \quad \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \in \mathbb{N}, \tag{1.4.4}$$

where $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$ correspond to the recurrence coefficients of $\{B_n\}_{n\in\mathbb{N}}$.

As a matter of fact, $J = s\big(\, h_a \circ \tau_b\,\big)$ (with $a, s \in \mathbb{C}^*$ and $b \in \mathbb{C}$) is the unique isomorphism which preserves the orthogonality of a sequence, as it was shown by Maroni [83].

# CHAPTER 2

## Classical orthogonal polynomials: some known and new results

The orthogonal polynomial sequences named as *Hermite*, *Laguerre* (with one parameter), *Bessel* (with one parameter) or *Jacobi* (with two parameters, including the special cases of *Gegenbauer, Legendre* and *Tchebyshev* polynomials) are collectively named as **classical orthogonal polynomials**. Just like the three musketeers[1], for a long period of time only three families of classical orthogonal polynomial sequences were known, but in 1949 Krall and Frink [66] gave to the Bessel polynomials the status of classical polynomials. The main reason of this late is related to the fact that the Bessel form is never positive-definite for any value of its parameter. From an algebraic point of view, an orthogonal polynomial sequence is said to be classical if its derivative sequence is also orthogonal:

**Definition 2.0.4** (Hahn's property [52])**.** The OPS $\{P_n\}_{n\geqslant 0}$ is **classical** when the sequence of derivatives $\{P_n^{[1]}\}_{n\geqslant 0}$ defined by (1.3.6) is also orthogonal. In this case, the corresponding regular form is said to be a **classical form**.

The classical polynomial sequences (or, loosely speaking, "classical polynomials") have been widely studied through years, so that a huge collection of their properties may be found in the literature, some of them will be stated here.

The definition adopted here for classical polynomials was originally presented by Hahn [52],

[1]This metaphor was suggested by Maroni [78].

but this property was also reached by Krall [61] and Webster [106], using different methods. As previously said, the classical polynomial sequences may be characterised through several ways. All the classical MOPS $\{P_n\}_{n\geqslant 0}$ have a number of properties in common of which the most important are listed below. Such properties characterise the classical polynomials, in the sense that any MOPS realising one of them can be reduced to a classical sequence. Thus, a given a MOPS $\{P_n\}_{n\geqslant 0}$ is classical if and only if one of the following properties is satisfied:

**Hahn's theorem:** [53] There exists $k \in \mathbb{N}^*$ such that $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is orthogonal.

**Classical functional equation:** (Geronimus [48]) There exist two polynomials $\Phi$ and $\Psi$ such that the corresponding regular form $u_0$ satisfies

$$D\big(\Phi u_0\big) + \Psi u_0 = 0\ , \tag{2.0.1}$$

where $\deg \Phi \leqslant 2$ ($\Phi$ monic) and $\deg(\Psi) = 1$.

**Bochner condition:** [18] There exist two polynomials, $\Phi$ monic, $\deg \Phi \leqslant 2$, $\Psi$, $\deg \Psi = 1$ and a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ with $\chi_0 = 0$ and $\chi_{n+1} \neq 0$, $n \in \mathbb{N}$, such that

$$\big(\mathcal{F}\, P_n\big)(x) = \chi_n P_n(x),\ \ n \in \mathbb{N}, \tag{2.0.2}$$

where

$$\mathcal{F} = \Phi(x)\, D^2 - \Psi(x)\, D\ . \tag{2.0.3}$$

**Rodrigues type formula:** [78, 82] There is a sequence of nonzero complex numbers $\{\vartheta_n\}_{n\in\mathbb{N}}$ and a monic polynomial $\Phi$ with $\deg \Phi \leqslant 2$ such that

$$P_n\, u_0 = \vartheta_n\, D^n(\Phi^n\, u_0)\ ,\quad n \in \mathbb{N} \qquad \text{(Rodrigues Formula)}. \tag{2.0.4}$$

**Structural relation:** [6] There exist a monic polynomial $\Phi$, with $\deg \Phi \leqslant 2$, and two polynomial sequences $\{C_n\}_{n\in\mathbb{N}}$, $\{D_n\}_{n\in\mathbb{N}}$ with $\deg C_n \leqslant 1$, $\deg D_{n+1} = 0$, for $n \in \mathbb{N}$, such that

$$\Phi(x)\, P'_{n+1}(x) = \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\,\big)P_{n+1}(x) - \gamma_{n+1}\, D_{n+1}\, P_n(x)\ , \tag{2.0.5}$$

holds for all $n \in \mathbb{N}$, with $\gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle}$, $n \in \mathbb{N}$.

An analogous relation to *Rodrigues* type formula, was also displayed by Maroni in [78] and it goes as follows:

*In order to a polynomial sequence* $\{P_n\}_{n\geqslant}$ *be classical, it is a necessary and sufficient condition to exist a sequence of nonzero complex numbers* $\{\varrho_n\}_{n\in\mathbb{N}}$ *and a monic polynomial* $\Phi$ *with* $\deg\Phi \leqslant 2$ *such that*

$$P_{n+1}u_0 = \varrho_n\, D\left(P_n^{[1]}\,\Phi\, u_0\right)\,, \quad n \geqslant 0. \tag{2.0.6}$$

Hahn's theorem is named after the work of Wolfgang Hahn, who, in 1938, was the first to put in evidence in a single page document [53] this property shared by the classical polynomials (which was also examined by Krall [62] and Webster [106]). Using the theory of linear forms, Maroni and da Rocha [86] gave a more instructive proof of this result.

After the works of Salomon Bochner [18] in 1929 and Krall and Frink [66] in 1949, it is known that the operator (2.0.3) has essentially (that is, up to a linear change of variable) four distinct OPS now known as classical sequences: Hermite, Laguerre, Bessel and Jacobi sequences.

Bochner has also implicitly imposed the problem of classifying all orthogonal polynomials satisfying the differential equation

$$L_N[y](x) := \sum_{i=0}^{N} l_i(x)\, y^{(i)}(x) = \lambda_n y(x)\,. \tag{2.0.7}$$

where $\{\lambda_n\}_{n\in\mathbb{N}}$ represent a sequence of real numbers. In 1938 Krall [65] gave a necessary and sufficient condition for an orthogonal polynomial set $\{B_n\}_{n\in\mathbb{N}}$ to satisfy a linear differential equation of the form (2.0.7). In particular, he has shown that if a linear differential operator $L_N[\cdot]$ has classical polynomials as eigenfunctions then it must be of even order, that is, $N = 2k$ for some $k \in \mathbb{N}$. A new proof of Krall's result was later given by Kwon et al. [69]. Later on, the same three authors improve the result of Krall by giving in [70] new results about the extension of Bochner result (see theorem 3.2 therein). Despite the interesting conditions found in the quoted works, an explicit and precise expression for the generalised equation is not given. On the other hand, Miranian [93] has shown that any even order differential operator having classical polynomials as eigenfunctions must be a polynomial with constant coefficients in the Bochner operator $\mathcal{F}$ given in (2.0.3). Once again, the methodology adopted was not constructive.

The section 2.2 of this chapter, is mainly concerned with the construction of an even order differential equation of type of (2.0.7) with $N = 2k$ having the classical polynomials as solutions. The structure of the polynomial coefficients $l_i(\cdot)$, $0 \leqslant i \leqslant 2k$, is thoroughly revealed (see theorem 2.2.1). The *modus operandi* of formal calculus is behind this construction. In theorem 2.2.3, we improve the results found in theorem 2.2.1 by giving explicit expressions for the polynomial coefficients $l_i(\cdot)$ instead of the recursively found previous ones. Subsequently, we expound how the even order differential operator $L_{2k}$ may be written as a polynomial with constant coefficients in the Bochner operator $\mathcal{F}$ and, conversely, how to express any power of $\mathcal{F}$ as a sum in $L_{2\tau}$ with $0 \leqslant \tau \leqslant k$. The bridge between these two operators can be done through the Stirling numbers. Therefore, in §2.2.3 we review this concept which is sufficient to study the cases of Hermite and Laguerre classical families, whereas the cases of Bessel or Jacobi sequences required the introduction of the concept of the so-called **A-modified Stirling numbers**, with $A$ representing a complex parameter. Based on these sets of numbers, we attain our first objective: to explicit establish a somewhat "inverse relation" between any power of $\mathcal{F}$ and $L_{2k}$. The analysis is guided separately for each classical family.

Concerning the reciprocal condition of Bochner's generalised differential equation, we bring a new proof, which we believe to shed new light to the theory (see theorem 2.4.1). At last, a generalisation of the Rodrigues type (functional) formula will come up with theorem 2.4.25 in §2.4.2.

## 2.1 Some other properties of the classical polynomial sequences

### 2.1.1 Invariance of the classical character by affine transformations

The classical character is invariant under any affine transformation $T = h_a \circ \tau_b$, with $a \in \mathbb{C}^*, b \in \mathbb{C}$, on $\mathcal{P}$. This is a direct consequence of $T$ being an isomorphism preserving the orthogonality. Precisely, if $u_0$ is a classical form satisfying the functional equation (2.0.1), then $\widetilde{u}_0 = \big(h_{a^{-1}} \circ \tau_{-b}\big)u_0$ is also classical and it satisfies the equation

$$D\left(\widetilde{\Phi}\,\widetilde{u}_0\right) + \widetilde{\Psi}\,\widetilde{u}_0 = 0,$$

with $\widetilde{\Phi}(x) = a^{-t}\,\Phi(ax+b)$, $\widetilde{\Psi}(x) = a^{1-t}\,\Psi(ax+b)$, where $t = \deg(\Phi) \leqslant 2$ [85].
Therefore it appears to be natural to define the following equivalence relation between forms [85]

$$\forall\, u, v \in \mathcal{P}', \quad u \sim v \quad \Leftrightarrow \quad \exists\, a \in \mathbb{C}^*,\, b \in \mathbb{C} \,:\, u = \big(h_{a^{-1}} \circ \tau_{-b}\big)v\,.$$

As a result, four equivalence classes arise essentially determined by the degree and the roots of the monic polynomial $\Phi$ (for this reason, also called *"leading" polynomial*) presented on (2.0.1), which are:

- Hermite forms $\mathcal{H}$, when $\deg\Phi = 0$ ;

- Laguerre forms $\mathcal{L}(\alpha)$, when $\deg\Phi = 1$ ;
- Bessel forms $\mathcal{B}(\alpha)$, when $\deg\Phi = 2$ and $\Phi$ has a single root;
- Jacobi forms $\mathcal{J}(\alpha,\beta)$, when $\deg\Phi = 2$ and $\Phi$ has two simple roots.

Under a convenient choice for the arbitrary parameters $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, it is possible to single out four canonical situations representative of the corresponding equivalence class. Hence, there will be no further consequences, if we take $\Phi(x) = 1$, $\Phi(x) = x$, $\Phi(x) = x^2$ and $\Phi(x) = x^2 - 1$ to be the representative choice for Hermite, Laguerre, Bessel and Jacobi classical families, respectively. Naturally, in the cases of Laguerre and Bessel one parameter will be undetermined and in the case of Jacobi family there will be two instead.

A detailed explanation may be followed in [85], but Table 2.1 resumes the information relative to the canonical classical forms by giving the polynomials $\Phi$ and $\Psi$ presented in (2.0.1) or in (2.0.3), the eigenvalues $\chi_n$, $n \in \mathbb{N}$, for the Bochner differential equation, the coefficients $\vartheta_n$, $n \in \mathbb{N}$, of the Rodrigues type formula, the sequences $\{C_n\}_{n\in\mathbb{N}}$, $\{D_n\}_{n\in\mathbb{N}}$ involved in the structure relation (2.0.5) and at last the recurrence coefficients $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$, of the associated classical sequence.

The conditions listed in the top line of Table 2.1 are the regularity conditions, in the sense that they must be satisfied otherwise the regularity of the classical form would be contradicted. A classical form of Hermite, $\mathcal{H}$, is always positive definite. On the opposite, as it was already said, the Bessel form $\mathcal{B}(\alpha)$ is never positive definite no matter the possible values of parameter $\alpha$ that guarantee the regularity of the form $\alpha \neq -\frac{n}{2}$, $n \in \mathbb{N}$. The form of Laguerre $\mathcal{L}(\alpha)$ is regular when $\alpha \neq -(n+1)$, for $n \in \mathbb{N}$, and it is positive definite if and only if $\alpha \in \mathbb{R}$ and $\alpha + 1 > 0$. Finally, a Jacobi form $\mathcal{J}(\alpha,\beta)$, with $\alpha,\beta \neq -n$ and $\alpha+\beta \neq -(n+1)$, for $n \in \mathbb{N}^*$, is positive definite if and only if $\alpha,\beta \in \mathbb{R}$ with $\alpha + 1 > 0$ and $\beta + 1 > 0$.

Table 2.1: Expressions for $\Phi$ and $\Psi$, $\chi_n$, $\vartheta_n$, $C_n$, $D_n$, with $n \in \mathbb{N}$, given in (2.0.1)-(2.0.5) and the corresponding recurrence coefficients $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$ for each classical family.

| | Hermite | Laguerre | Bessel | Jacobi |
|---|---|---|---|---|
| | $\mathcal{H}$ | $\mathcal{L}(\alpha)$ | $\mathcal{B}(\alpha)$ | $\mathcal{J}(\alpha,\beta)$ |
| with $n \in \mathbb{N}$ | | $\alpha \neq -(n+1)$ | $\alpha \neq -\frac{n}{2}$ | $\alpha,\beta \neq -(n+1)$<br>$\alpha+\beta \neq -(n+2)$ |
| $\Phi(x)$ | $1$ | $x$ | $x^2$ | $x^2-1$ |
| $\Psi(x)$ | $2x$ | $x-\alpha-1$ | $-2\,(\alpha x+1)$ | $-(\alpha+\beta+2)\,x+(\alpha-\beta)$ |
| $\chi_n$ | $-2n$ | $-n$ | $n(n+2\alpha-1)$ | $n(n+\alpha+\beta+1)$ |
| $\vartheta_n$ | $(-2)^{-n}$ | $(-1)^n$ | $\frac{\Gamma(n+2\,\alpha-1)}{\Gamma(2n+2\,\alpha-1)}$ | $\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}$ |
| $C_n$ | $-2x$ | $-x+2n+\alpha$ | $2(n+\alpha-1)x+\frac{2(\alpha-1)}{n+\alpha-1}$ | $(2n+\alpha+\beta)x-\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)}$ |
| $D_n$ | $-2$ | $-1$ | $2n+2\alpha-1$ | $2n+\alpha+\beta+1$ |
| $\beta_n$ | $0$ | $2n+\alpha+1$ | $\frac{1-\alpha}{(n+\alpha-1)(n+\alpha)}$ | $\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$ |
| $\gamma_{n+1}$ | $\frac{n+1}{2}$ | $(n+1)(n+\alpha+1)$ | $\frac{-(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}$ | $\frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}$ |

### 2.1.2 Invariance of the classical character by differentiation

The classical character of a form does not only remain invariant under any affine transformation but also under a differentiation of any order. Regarding the importance of this for the sequel, we recall this result formally.

**Corollary 2.1.1.** [78, 82] *If the MOPS $\{P_n\}_{n\in\mathbb{N}}$ is classical, then so is $\{P_n^{[k]}\}_{n\in\mathbb{N}}$, whenever $k \geqslant 1$, and any polynomial $P_{n+1}^{[k]}$ fulfils the following differential equation:*

$$\Phi\left(P_n^{[k]}\right)'' - (\Psi - k\,\Phi')\left(P_n^{[k]}\right)' = \chi_n^{[k]}\left(P_n^{[k]}\right), \quad n \in \mathbb{N}, \tag{2.1.1}$$

*where $\Phi,\ \Psi \in \mathcal{P}$ (with $\Phi$ monic and $\deg\Phi \leqslant 2,\ \deg\Psi = 1$) and $\chi_0^{[k]} = 0$,*

$$\chi_{n+1}^{[k]} = (n+1)\left\{\frac{n+2k}{2}\,\Phi''(0) - \Psi'(0)\right\} \neq 0,\ n \in \mathbb{N}.$$

*The corresponding classical forms are related by the equality:*

$$u_0^{[k]} = \zeta_k\,\Phi^k\,u_0\,, \tag{2.1.2}$$

*for some $\zeta_k \neq 0$.*

The previous result asserts that the sequence of normalised derivatives of a given classical sequence is still a classical polynomial sequence, belonging to the same class. Before going any further, we shall remark an important consequence. All the properties of the normalised derivatives of a classical sequence may be managed without making a single differentiation: if $\{H_n\}_{n\geqslant 0}$, $\{L_n(\cdot;\alpha)\}_{n\geqslant 0}$, $\{B_n(\cdot;\alpha)\}_{n\geqslant 0}$ and $\{J_n(\cdot;\alpha,\beta)\}_{n\geqslant 0}$ represent, respectively, the Hermite, Laguerre, Bessel and Jacobi polynomials, we then have for a given positive integer $k$:

$$\begin{aligned} &H_n^{[k]}(x) = H_n(x) && L_n^{[k]}(x;\alpha) = L_n(x;\alpha+k) \\ &B_n^{[k]}(x;\alpha) = B_n(x;\alpha+k) && J_n^{[k]}(x;\alpha,\beta) = J_n(x;\alpha+k,\beta+k), \end{aligned} \tag{2.1.3}$$

for $n \in \mathbb{N}^*$. Notice that the previous relations presume the parameters $\alpha$ or $\beta$ to take values in $\mathbb{C}$ within the range of regularity, which has been already mentioned at the top line of Table 2.1.

## 2.2 New results on Bochner differential equation

During this section we will be dealing with the construction of an even order differential equation of the type (2.0.7) with $N = 2k$ for some $k \in \mathbb{N}$. According to Hahn's theorem, a

MOPS $\{P_n\}_{n\in\mathbb{N}}$ is said to be classical whenever $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is also orthogonal. This will be on focus throughout this section. For the sake of simplicity, whenever there is no danger of confusion, we will adopt the notation $Q_n := P_n^{[k]}$ with $k \in \mathbb{N}^*$ and the elements of the dual sequence associated to $\{Q_n\}_{n\in\mathbb{N}}$ will be denoted as $v_n$, instead of $u_n^{[k]}$, $n \in \mathbb{N}$ as previously suggested.

### 2.2.1 Generalised Bochner differential equation

As claimed before, the construction of an even $(2k)$ order linear differential equation with polynomial coefficients recursively define having classical polynomials as eigenfunctions is in the pipeline.

**Theorem 2.2.1.** *[73] Let $\{P_n\}_{n\in\mathbb{N}}$ be an OPS. If there is an integer $k \geqslant 1$ such that the MPS $\{Q_n\}_{n\in\mathbb{N}}$ is also orthogonal, then any polynomial $P_{n+k}$ fulfils the following differential equation of order $2k$:*

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x)\; D^{k+\nu} P_{n+k}(x) = \Xi_n(k) P_{n+k}(x), \; n \in \mathbb{N}, \tag{2.2.1}$$

*where*

$$\Lambda_\nu(k;x) = \frac{1}{\nu!}\sum_{\mu=0}^{\nu} \lambda_\mu^k \Omega_{\nu-\mu}^k(\nu;x) P_{k+\mu}(x), \quad 0 \leqslant \nu \leqslant k; \tag{2.2.2}$$

$$\Xi_n(k) = \lambda_n^k \left(n+1\right)_k, \quad n \in \mathbb{N}; \tag{2.2.3}$$

$$\lambda_n^k = (-1)^k \frac{\langle\, v_0, {Q_n}^2 \,\rangle}{\langle\, u_0, P_{n+k}^2 \,\rangle} \left(n+1\right)_k, \quad n \in \mathbb{N}; \tag{2.2.4}$$

*and*

$$\begin{cases} \Omega_0^k(0;\cdot) = 1, \\ \Omega_0^k(\mu+1;\cdot) = 1, \quad \mu \in \mathbb{N}, \\ \Omega_{\mu+1-\xi}^k(\mu+1;\cdot) = -\sum\limits_{\nu=\xi}^{\mu} \dfrac{1}{\nu!} \left(Q_{\mu+1}\right)^{(\nu)} \Omega_{\nu-\xi}^k(\nu;\cdot), \quad 0 \leqslant \xi \leqslant \mu, \end{cases} \tag{2.2.5}$$

*with $(n+1)_k$ represents the Pochhammer symbol defined in (**??**).*

The Bochner equation (2.0.2) comes as a particular case of the achieved equation (upon the particular choice of $k = 1$). When we consider $k = 2$ we recover the fourth order differential equation achieved by Maroni [82] (see §7 therein) and also by Lesky [71].

*Proof.* Supose $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ to be two MOPS. According to (1.4.2), the elements of their dual sequences satisfy the relations

$$u_n = \left(\langle u_0, {P_n}^2\rangle\right)^{-1} P_n u_0, \quad n \geqslant 0, \tag{2.2.6}$$

$$v_n = \left(\langle v_0, {Q_n}^2\rangle\right)^{-1} Q_n v_0, \quad n \geqslant 0. \tag{2.2.7}$$

Recalling the considerations made on page 27, the equality (1.3.10) holds true, which, after (2.2.6)-(2.2.7), becomes:

$$(Q_n\, v_0)^{(k)} = \lambda_n^k\, P_{n+k} u_0, \quad n \in \mathbb{N}, \tag{2.2.8}$$

with

$$\lambda_n^k = (-1)^k \frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+k}^2\rangle} \prod_{\mu=1}^{k} (n+\mu)\ , \quad n \in \mathbb{N}. \tag{2.2.9}$$

By virtue of the Leibniz relation, the first member of (2.2.8) may be written as

$$(Q_n\, v_0)^{(k)} = \sum_{\nu=0}^{k} \binom{k}{\nu} (Q_n)^{(\nu)} (v_0)^{(k-\nu)}, \quad n \in \mathbb{N}, \tag{2.2.10}$$

which allows to transform (2.2.8) into

$$\sum_{\nu=0}^{k} \binom{k}{\nu} (Q_n)^{(\nu)} (v_0)^{(k-\nu)} = \lambda_n^k\, P_{n+k}\, u_0, \quad n \in \mathbb{N}. \tag{2.2.11}$$

Whenever $\nu \geqslant n+1$, $(Q_n)^{(\nu)} = 0$, so from the previous we have

$$\sum_{\nu=0}^{n} \binom{k}{\nu} (Q_n)^{(\nu)} (v_0)^{(k-\nu)} = \lambda_n^k\, P_{n+k}\, u_0, \quad 0 \leqslant n \leqslant k. \tag{2.2.12}$$

In particular, taking $n=0$, we get

$$(v_0)^{(k)} = \lambda_0^k P_k u_0. \tag{2.2.13}$$

Similarly, if we consider $n = 1$ in (2.2.12), then, on account of the precedent equality, we obtain

$$k\, (v_0)^{(k-1)} = \left(\lambda_1^k P_{k+1} - \lambda_0^k Q_1 P_k\right) u_0. \tag{2.2.14}$$

Let us now suppose there is a set of polynomials $\{\Omega_\tau^k(\nu;\cdot) : 0 \leqslant \tau \leqslant \nu\}_{0\leqslant\nu\leqslant k}$ allowing to express

$$\frac{k!}{(k-\nu)!} (v_0)^{(k-\nu)} = \left(\sum_{\zeta=0}^{\nu} \lambda_\zeta^k \Omega_{\nu-\zeta}^k (\nu; x)\, P_{k+\zeta}(x)\right) u_0, \quad 0 \leqslant \nu \leqslant \mu < k, \tag{2.2.15}$$

and such that

$$\Omega_0^k(\nu;x)=1\ .$$

The equalities (2.2.13) and (2.2.14) provide

$$\begin{aligned}&\Omega_0^k(0;x)=1,\\&\Omega_1^k(1;x)=-Q_1(x)\,,\qquad \Omega_0^k(1;x)=1.\end{aligned}\tag{2.2.16}$$

The expression (2.2.12) with $n$ replaced by $\mu+1$ becomes

$$\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)}=\lambda_{\mu+1}^kP_{\mu+1+k}u_0-\sum_{\nu=0}^{\mu}\binom{k}{\nu}(Q_{\mu+1})^{(\nu)}(v_0)^{(k-\nu)}\,.$$

Taking into account the assumption (2.2.15), it yields from the previous

$$\begin{aligned}\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)}\ &=\ \Big[\lambda_{\mu+1}^kP_{\mu+1+k}(x)\\&-\sum_{\nu=0}^{\mu}\sum_{\zeta=0}^{\nu}\frac{1}{\nu!}(Q_{\mu+1}(x))^{(\nu)}\lambda_\zeta^k\Omega_{\nu-\zeta}^k(\nu;x)P_{k+\zeta}(x)\Big]u_0,\end{aligned}$$

which may be expressed as

$$\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)}=\Big[\lambda_{\mu+1}^kP_{\mu+1+k}-\sum_{\zeta=0}^{\mu}\lambda_\zeta^kP_{k+\zeta}\sum_{\nu=\zeta}^{\mu}\frac{1}{\nu!}(Q_{\mu+1})^{(\nu)}\Omega_{\nu-\zeta}^k(\nu;\cdot)\Big]u_0.$$

This last relation is read as

$$\frac{k!}{(k-\mu-1)!}(v_0)^{(k-\mu-1)}=\sum_{\zeta=0}^{\mu+1}\lambda_\zeta^k\Omega_{\mu+1-\zeta}^k(\mu+1;\cdot)P_{k+\zeta}u_0,\tag{2.2.17}$$

by virtue of (2.2.5). Substituting $(v_0)^{(k-\nu)}$ given by (2.2.15) into (2.2.11), we obtain

$$\sum_{\nu=0}^{k}\binom{k}{\nu}(Q_n)^{(\nu)}(x)\frac{(k-\nu)!}{k!}\left(\sum_{\zeta=0}^{\nu}\lambda_\zeta^k\Omega_{\nu-\zeta}^k(\nu;x)P_{k+\zeta}(x)u_0\right)=\lambda_n^kP_{n+k}u_0,\quad n\geqslant 0,$$

or, by reordering the terms, we get

$$\sum_{\nu=0}^{k}\frac{1}{\nu!}\left(\sum_{\zeta=0}^{\nu}\lambda_\zeta^k\Omega_{\nu-\zeta}^k(\nu;x)P_{k+\zeta}(x)\right)(Q_n)^{(\nu)}u_0=\lambda_n^kP_{n+k}u_0,\quad n\geqslant 0.$$

Based on the property of the regular form $u_0$ shown in lemma 1.4.3, the previous relation implies

$$\sum_{\nu=0}^{k}\frac{1}{\nu!}\left(\sum_{\zeta=0}^{\nu}\lambda_\zeta^k\Omega_{\nu-\zeta}^k(\nu;x)P_{k+\zeta}(x)\right)(Q_n)^{(\nu)}=\lambda_n^kP_{n+k}\,,\quad n\geqslant 0.\tag{2.2.18}$$

Since

$$(Q_n)^{(\nu)}(x) = \left( \prod_{\mu=1}^{k} (n+\mu) \right)^{-1} (P_{n+k})^{(k+\nu)}(x) \ , \ \nu \geqslant 0, \tag{2.2.19}$$

we obtain (2.2.1)-(2.2.3). □

Concerning the polynomial coefficients $\Lambda_\nu(k;\cdot)$ presented in the differential equation (2.2.1), there are some of considerations to be made. In due course, a more powerful result providing their explicit expressions will come out.

**Remark 2.2.1.** The polynomials $\Lambda_i$, $i = 0, 1, 2$, defined in (2.2.2) may be expressed as follows:

$$\begin{aligned} \Lambda_0(k;x) &= \lambda_0^k P_k(x), \\ \Lambda_1(k;x) &= E_k(x) P_{k+1}(x) + F_k(x) P_k(x) \\ \Lambda_2(k;x) &= G_k(x) P_{k+1}(x) + H_k(x) P_k(x) \end{aligned}$$

where

$$E_k(x) = \lambda_1^k \ ,$$

$$F_k(x) = -\lambda_0^k Q_1(x) \ ,$$

$$G_k(x) = \frac{1}{2}\left\{ -\lambda_1^k Q_2'(x) + \lambda_2^k (x - \beta_{k+1}) \right\} \ ,$$

$$H_k(x) = \frac{1}{2}\left\{ \lambda_0^k \left( -Q_2(x) + Q_2'(x) Q_1(x) \right) - \lambda_2^k \gamma_{k+1} \right\}.$$

Naturally, for $k \geqslant 1$, $\deg(E_k) = 0$, $\deg(F_k) = 1$, $\deg(G_k) \leqslant 1$ and $\deg(H_k) = 2$.

It appears to be important to know more about the degree of the $\Lambda$-polynomials given in (2.2.2). Once this depends on the degree of $\Omega$-polynomials presented in (2.2.5), we are obliged to analyze these elements in first place.

**Lemma 2.2.2.** *[73] The polynomials $\Omega_\mu^k(\nu,\cdot)$ have degree $\mu$, $0 \leqslant \mu \leqslant \nu$; precisely*

$$\Omega_\mu^k(\nu;x) = (-1)^\mu \binom{\nu}{\nu-\mu} x^\mu + \ldots, \quad 0 \leqslant \mu \leqslant \nu \ . \tag{2.2.20}$$

*Consequently, we have the following results:*

- *for Hermite and Laguerre cases,*

$$\begin{aligned} &\deg \Lambda_0(k;x) = k, \\ &\deg \Lambda_\nu(k;x) \leqslant \nu + k - 1, \ \nu \geqslant 1; \end{aligned} \tag{2.2.21}$$

- *for Bessel and Jacobi cases,*

$$\begin{aligned} &\deg \Lambda_\nu(k;x) = k+\nu,\ 0 \leqslant \nu \leqslant k,\\ &\deg \Lambda_\nu(k;x) \leqslant \nu+k-1,\ \nu \geqslant k+1. \end{aligned} \tag{2.2.22}$$

*Proof.* Writing $\Omega_\mu^k(\nu;x) = \omega_\mu^k(\nu)x^\mu + \dots$ , from (2.2.5) and (2.2.19), we easily obtain

$$\omega_{\mu+1-\xi}^k(\mu+1) = -\sum_{\nu=\xi}^{\mu} \binom{\mu+1}{\nu} \omega_{\nu-\xi}^k(\nu), \quad 0 \leqslant \xi \leqslant \mu\,. \tag{2.2.23}$$

Now, taking $\xi = \mu$, we have

$$\omega_1^k(\mu+1) = -(\mu+1)\omega_0^k(\mu) = -\binom{\mu+1}{\mu}, \quad \mu \geqslant 0,$$

since $\omega_0^k(\mu) = 1,\ \mu \geqslant 0$, according to the definition. When $\xi = \mu-1$, for $\mu \geqslant 1$, we obtain from (2.2.23)

$$\omega_2^k(\mu+1) = \binom{\mu+1}{\mu-1} \quad .$$

Let us take $\xi = \mu-\tau$, $0 \leqslant \tau \leqslant \mu$. The relation (2.2.23) can be read as

$$\omega_{\tau+1}^k(\mu+1) = -\sum_{\zeta=0}^{\tau} \binom{\mu+1}{\mu-\tau+\zeta} \omega_\zeta^k(\mu-\tau+\zeta)$$

which admits the representation

$$\omega_{\tau+1}^k(\mu+1) = -\binom{\mu+1}{\mu-\tau} - \sum_{\zeta=0}^{\tau-1} \binom{\mu+1}{\mu+1-\tau+\zeta} \omega_{\zeta+1}^k(\mu+1-\tau+\zeta)\,. \tag{2.2.24}$$

Under the assumption $\omega_{\tau+1}^k(\mu) = (-1)^{\tau+1}\binom{\mu}{\mu-1-\tau}$, with $\tau+1 \leqslant \mu$, the equality (2.2.24) may be transformed into

$$\begin{aligned} \omega_{\tau+1}^k(\mu+1) &= -\sum_{\nu=\mu-\tau}^{\mu} \binom{\mu+1}{\nu} (-1)^{\nu-(\mu-\tau)} \binom{\nu}{\mu-\tau} \\ &= -\binom{\mu+1}{\mu-\tau} - \sum_{\zeta=0}^{\tau-1} (-1)^{\zeta+1} \binom{\mu+1}{\mu+1-\tau+\zeta} \binom{\mu+1-\tau+\zeta}{\zeta+1} \\ &= (-1)^{\tau+1} \binom{\mu+1}{\mu-\tau}. \end{aligned}$$

Consequently, (2.2.20) holds. Now, from (2.2.2) and (2.2.20), we have

$$\Lambda_\nu(k;x) = \left\{ \frac{1}{\nu!} \sum_{\mu=0}^{\nu} \lambda_\mu^k \, (-1)^{\nu-\mu} \binom{\nu}{\mu} \right\} x^{k+\nu} + \dots$$

Following the definition of the recurrence coefficients of an orthogonal sequence, we have

$$\langle u_0, P_{n+1}^2 \rangle = \prod_{\nu=0}^{n} \gamma_{\nu+1}, \quad n \geqslant 0.$$

Therefore, taking into account (2.1.3) and the surrounding considerations, these last equalities permit to deduce the explicit expression of the coefficients $\lambda_n^k$, with $n \geqslant 0$, defined in (2.2.4), for each classical family.

For the Hermite and Laguerre cases, the coefficients $\lambda_n^k$ do not depend on $n$, since they are respectively given by

$$\lambda_n^k = (-2)^k \quad \text{(Hermite)} \tag{2.2.25}$$

and

$$\lambda_n^k = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+k)} \quad \text{(Laguerre)}\ , \tag{2.2.26}$$

therefore (2.2.21) holds.

In the Bessel case, we easily obtain

$$\lambda_n^k = C_\alpha^k \frac{\Gamma(2\alpha-1+2k+n)}{\Gamma(2\alpha-1+k+n)} \tag{2.2.27}$$

with

$$C_\alpha^k = \frac{4^{-k}\Gamma(2\alpha+2k)}{\Gamma(2\alpha)}\,.$$

Consider

$$\Lambda_\nu(k;x) = C_\alpha^k \frac{1}{\nu!}\, b_\nu^k(\alpha)\, x^{k+\nu} + \ldots$$

with

$$b_\nu^k(\alpha) = \sum_{\mu=0}^{\nu} (-1)^{\nu-\mu} \binom{\nu}{\mu} \frac{\Gamma(2\alpha-1+2k+\mu)}{\Gamma(2\alpha-1+k+\mu)}\,.$$

After some calculations, we get

$$\frac{b_{\nu+1}^k(\alpha)}{b_\nu^k(\alpha)} = -\frac{\nu-k}{\nu+2\alpha-1+k}\,,\ \nu \geqslant 0\,.$$

It follows $b_\nu^k(\alpha) = 0,\ \nu \geqslant k+1$, and

$$b_\nu^k(\alpha) = b_0(\alpha)\, \frac{\Gamma(k+\nu)}{\Gamma(k)}\, \frac{\Gamma(2\alpha-1+k)}{\Gamma(2\alpha-1+k+\nu)}\,,\ 0 \leqslant \nu \leqslant k\,.$$

In the Jacobi case, we have

$$\lambda_n^k = C_\alpha^k \frac{\Gamma(\alpha+\beta+1+2k+n)}{\Gamma(\alpha+\beta+1+k+n)} \tag{2.2.28}$$

with

$$C_\alpha^k = \frac{(-4)^{-k}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+2+2k)}{\Gamma(\alpha+1+k)\Gamma(\beta+1+k)\Gamma(\alpha+\beta+2)}\,.$$

With analogous results as above, we finally obtain (2.2.22). □

The information concerning the expressions of the coefficients $\lambda_n^k$ for each classical family is summarised in the following table.

Table 2.2: Expressions for $\lambda_n(k)$, with $n \in \mathbb{N}$, for each classical family. (Note the regularity conditions already mentioned in Table 2.1 )

| | Hermite | Laguerre | Bessel | Jacobi |
|---|---|---|---|---|
| $\lambda_n(k)$ | $(-2)^k$ | $\frac{(-1)^k}{(\alpha+1)_k}$ | $C_\alpha^k\,(2\alpha-1+k+n)_k$ | $C_{\alpha,\beta}^k\,(\alpha+\beta+k+n+1)_k$ |
| with | | | $C_\alpha^k = 4^{-k}\,(2\alpha)_{2k}$ | $C_{\alpha,\beta}^k = \frac{(-4)^{-k}\,(\alpha+\beta+2)_{2k}}{(\alpha+1)_k\,(\beta+1)_k}$ |

Through the implementation of the recurrence relation for the polynomials $\Lambda_\nu(k;\cdot)$ in a symbolic computational language like *Mathematica*©, Loureiro et al. [73] have presented the differential equation (2.2.2) for the first values of $k$ ($k=1,2,3$) and for each classical family (*cf.* §4 therein). The explicit determination of the polynomials $\Lambda_\nu(k;\cdot)$ for any $0 \leqslant \nu \leqslant k$ is far from being obvious if one tries to solve the recurrence relation that they fulfil, however it becomes easier if we use another methodology to obtain a $2k$-order differential equation and then show that the polynomial coefficients presented there in are the same as the ones in (2.2.2). This is a brief sketch of the next result.

**Theorem 2.2.3.** *[75] Under the same assumptions of theorem 2.2.1, there is a monic polynomial $\Phi$ with $\deg\Phi \leqslant 2$, and a one-degree polynomial $\Psi$, such that the polynomials $\Lambda_\nu(k;\cdot)$, with $0 \leqslant \nu \leqslant k$, given by (2.2.2) may also be expressed by:*

$$\Lambda_\nu(k;x) = \frac{\lambda_0^k\;\omega_{k,\nu}}{\nu!}\;\Phi^\nu(x)\;\Big(P_k(x)\Big)^{(\nu)},\quad 0 \leqslant \nu \leqslant k, \tag{2.2.29}$$

*with*

$$\omega_{k,\nu} = \begin{cases} \big(-\Psi'(0)\big)^{-\nu} & \text{if}\quad 0 \leqslant \deg\Phi \leqslant 1\,, \\ \dfrac{1}{\big(\,k-1-\Psi'(0)\,\big)_\nu} & \text{if}\quad \deg\Phi = 2\,, \end{cases} \tag{2.2.30}$$

*where the elements of the two nonzero sequences $\{\lambda_n^k\}_{n\in\mathbb{N}}$ and $\{\,\Xi_n(k)\,\}_{n\in\mathbb{N}}$, are respectively given in (2.2.4) and (2.2.3).*

*Proof.* Under the same assumptions of theorem 2.2.1, as previously determined in the corresponding proof the relation (2.2.11) with $\lambda_n^k$ given by (2.2.9).

The fact that $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ are both orthogonal provides the classical character of $\{P_n\}_{n\in\mathbb{N}}$, so there exist a monic polynomial $\Phi$ and a polynomial $\Psi$, with $\deg\Phi \leqslant 2$ and $\deg\Psi = 1$, such that the regular form $u_0$ fulfils (2.0.1). By virtue of corollary 2.1.1, $\{Q_n\}_{n\in\mathbb{N}}$ is also classical and the associated classical form $v_0$ satisfies the equality $v_0 = \zeta_k\,\Phi^k\,u_0$, where $\zeta_k$ represents a nonzero constant. Also for any positive integer $j$ $\{P_n^{[j]}\}_{n\in\mathbb{N}}$ is a classical MOPS fulfilling the differential equation (2.1.1) with $k$ replaced by $j$ and $n$ by $n+1$, which may be expressed as follows

$$\Phi(x)\left(P_n^{[j+1]}(x)\right)' - \big\{\Psi(x) - j\,\Phi'(x)\big\}P_n^{[j+1]}(x) = \widetilde{\chi}_{n,j}P_{n+1}^{[j]}(x),\ 1\leqslant j\leqslant k,\ n\in\mathbb{N}. \quad (2.2.31)$$

with $\widetilde{\chi}_{n,j} = \frac{n+2j}{2}\Phi''(0) - \Psi'(0)$, $n\in\mathbb{N}$, because $\left(P_{n+1}^{[j]}\right)' = (n+1)\,P_n^{[j+1]}$, $n\in\mathbb{N}$.

By differentiating both members of $v_0 = \zeta_k\Phi^k\,u_0$ and then taking into consideration (2.0.1), we obtain the identity

$$(v_0)' = \zeta_k\,\big\{(k-1)\Phi'\,\Phi^{k-1}\,u_0 - \Phi^{k-1}\,\Psi\,u_0\big\},$$

which, because of (2.2.31) with $n=0$ and $j=k-1$, may be written like

$$(v_0)' = \zeta_k\,\Phi^{k-1}\big\{(k-1)\Phi''(0)\ -\Psi(0)\big\}\,P_1^{[k-1]}\,u_0.$$

By finite induction, it might be achieved that

$$\big(v_0\big)^{(j)} = \zeta_k\,\left(\prod_{\tau=1}^{j}\widetilde{\chi}_{\tau-1,k-\tau}\right)\,P_j^{[k-j]}\,\Phi^{k-j}\,u_0\,,\quad 1\leqslant j\leqslant k, \qquad (2.2.32)$$

where

$$\widetilde{\chi}_{\mu,\sigma} = \frac{\mu+2\sigma}{2}\,\Phi''(0) - \Psi'(0)\,,\quad \mu,\sigma\in\mathbb{N}.$$

Indeed, a single differentiation of both members of (2.2.32) leads to

$$\begin{aligned}(v_0)^{(j+1)} &= \zeta_k\,\left(\prod_{\tau=1}^{j}\widetilde{\chi}_{\tau-1,k-\tau}\right)\,\Big\{\left(P_j^{[k-j]}\right)'\,\Phi^{k-j}\,u_0 \\ &\quad +P_j^{[k-j]}\,\Big((k-j-1)\Phi'\,\Phi^{k-j-1}\,u_0 + \Phi^{k-j-1}\big(\Phi\,u_0\big)'\,\Big)\Big\}\end{aligned}$$

which, on account (2.0.1), becomes :

$$\begin{aligned}&(v_0)^{(j+1)} \\ &= \zeta_k\left\{\prod_{\tau=1}^{j}\widetilde{\chi}_{\tau-1,k-\tau}\right\}\Phi^{k-j-1}\Big\{\Phi\,\left(P_j^{[k-j]}\right)' + \Big((k-j-1)\Phi'\ -\Psi\,\Big)\,P_j^{[k-j]}\Big\}u_0.\end{aligned}$$

By virtue of (2.2.31) with the pair $(n,j)$ replaced by $(j,k-j-1)$, we deduce that the previous identity corresponds to (2.2.32) with $j+1$ instead of $j$, whence we conclude that (2.2.32) is valid for each positive integer $j$. In particular, when $j=k$, (2.2.32) becomes

$$(v_0)^{(k)} = \zeta_k \left( \prod_{\tau=1}^{k} \widetilde{\chi}_{\tau-1,k-\tau} \right) P_k \, u_0 \, . \tag{2.2.33}$$

On the other hand, if we consider $n=0$ in (2.2.11) we also obtain

$$(v_0)^{(k)} = \lambda_0^k \, P_k \, u_0 \, . \tag{2.2.34}$$

From the comparison between (2.2.33) and (2.2.34) we achieve the conclusion:

$$\zeta_k = \left( \prod_{\tau=1}^{k} \widetilde{\chi}_{\tau-1,k-\tau} \right)^{-1} \lambda_0^k \, .$$

Bringing this information into (2.2.32) with $j$ replaced by $k-\nu$, we obtain:

$$(v_0)^{(k-\nu)} = \omega_{k,\nu} \, \lambda_0^k \, \Phi^\nu \, P_{k-\nu}^{[\nu]} \, u_0 \tag{2.2.35}$$

where

$$\omega_{k,\nu} = \begin{cases} \left( \displaystyle\prod_{\tau=k-\nu+1}^{k} \widetilde{\chi}_{\tau-1,k-\tau} \right)^{-1} & , \quad 1 \leqslant \nu \leqslant k \\ 1 & , \quad \nu = 0 \, . \end{cases}$$

Based on the definition of $\widetilde{\chi}_{k-\tau-1,\tau}$, the coefficients $\omega_{k,\nu}$ may be expressed like:

$$\omega_{k,\nu} = \begin{cases} \left[ \displaystyle\prod_{\tau=0}^{\nu-1} \left( \frac{k+\tau-1}{2} \, \Phi''(0) - \Psi'(0) \right) \right]^{-1} & , \quad 1 \leqslant \nu \leqslant k \\ 1 & , \quad \nu = 0 \, , \end{cases}$$

Since $\Phi$ is a monic polynomial with $\deg \Phi \leqslant 2$, then, recalling (**??**), $\omega_{k,\nu}$ may be expressed as in (2.2.30). Hence, on account (2.2.35), the relation (2.2.11) may be transformed into

$$\sum_{\nu=0}^{k} \binom{k}{\nu} (Q_n)^{(\nu)} \; \lambda_0^k \, \omega_{k,\nu} \, P_{k-\nu}^{[\nu]} \, \Phi^\nu \, u_0 = \lambda_n^k \, P_{n+k} \, u_0 \; , \; n \in \mathbb{N}. \tag{2.2.36}$$

which, based on the regularity of $u_0$, provides

$$\sum_{\nu=0}^{k} \binom{k}{\nu} \left\{ \lambda_0^k \, \omega_{k,\nu} \, P_{k-\nu}^{[\nu]}(x) \, \Phi^\nu(x) \right\} \big( Q_n(x) \big)^{(\nu)} = \lambda_n^k \, P_{n+k}(x) \; , \; n \in \mathbb{N}.$$

Due to (2.2.19) the precedent equalities correspond to

$$\sum_{\nu=0}^{k} \widehat{\Lambda}_\nu(k;x)\ D^{k+\nu}\big(P_{n+k}(x)\big) = \Xi_n(k)\ P_{n+k}(x)\ ,\ n \in \mathbb{N}. \tag{2.2.37}$$

where

$$\widehat{\Lambda}_\nu(k;x) = \binom{k}{\nu}\ \lambda_0^k\ \omega_{k,\nu}\ P_{k-\nu}^{[\nu]}(x)\ \Phi^\nu(x),\ 0 \leqslant \nu \leqslant k,$$

and $\Xi_n(k)$ is given by (2.2.3). Clearly, under the definition of $P_{k-\nu}^{[\nu]}(\cdot)$, we easily observe that

$$\widehat{\Lambda}_\nu(k;x) = \frac{\lambda_0^k\ \omega_{k,\nu}}{\nu!}\ \Phi^\nu(x)\ \Big(P_k(x)\Big)^{(\nu)},\quad 0 \leqslant \nu \leqslant k.$$

Now, comparing (2.2.1) with (2.2.37) and representing by

$$A_\nu(k;x) = \Lambda_\nu(k;x) - \widehat{\Lambda}_\nu(k;x)\ ,\quad 0 \leqslant \nu \leqslant k\ ,$$

we deduce that

$$\sum_{\nu=0}^{k} A_\nu(k;x)\ D^{k+\nu}(P_{n+k}) = 0\ ,\quad n \in \mathbb{N}\ .$$

Since $D^{k+\nu}\big(P_j(x)\big) = 0,\ 0 \leqslant j \leqslant k-1$, it is obvious that

$$\sum_{\nu=0}^{k} A_\nu(k;x)\ D^{k+\nu}(P_n) = 0\ ,\quad n \in \mathbb{N}\ .$$

Based on the fact that $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathcal{P}$, we conclude from the previous equalities that

$$\sum_{\nu=0}^{k} A_\nu(k;x)\ D^{k+\nu} f = 0\ ,\quad f \in \mathcal{P}. \tag{2.2.38}$$

The particular choice $f(x) = x^k$ in (2.2.38) provides $A_0(k;\cdot) = 0$. Let us suppose that

$$A_\nu(k;\cdot) = 0,\ 0 \leqslant \nu \leqslant \mu \leqslant k-1.$$

If we consider $f(x) = x^{k+\mu+1}$ in (2.2.38), then, under the assumption, we easily derive that

$$A_{\mu+1}(k;x)\ (k+\mu+1)! = 0$$

which implies $A_{\mu+1}(k;x) = 0,\ 0 \leqslant \mu \leqslant k-1$. Therefore $A_\nu(k;x) = 0,\ 0 \leqslant \nu \leqslant k$, whence the result. □

**Remark 2.2.2.** Consider $\{P_n\}_{n\in\mathbb{N}}$ to be a classical MOPS. By virtue of Hahn's theorem (stated on page 32), there exists $k \geqslant 1$ such that $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is a MOPS, whence, if $\tau$ is an integer between $1$ and $k$, $\{P_n^{[\tau]}\}_{n\in\mathbb{N}}$ is also orthogonal. Therefore from theorem 2.2.1, we deduce that $P_n$ still fulfils the differential equation (2.2.1) with the pair $(n,k)$ replaced by $(n-\tau,\tau)$ and $n \geqslant \tau$.
It can be easily seen that when $0 \leqslant n \leqslant \tau-1$, necessarily $D^{\tau+\nu}(P_n)=0$ (with $0\leqslant \nu \leqslant \tau$) and $\Xi_{n-\tau}(\tau)=0$. This last equality is related to the fact that $\{n\}_{(\tau)} = (n-\tau+1)_\tau = 0$ when $0\leqslant n \leqslant \tau-1$ (it is a simple consequence of the definition of the falling factorial of a number (**??**) ). This allows us to conclude that each element of $\{P_n\}_{n\in\mathbb{N}}$ is also a solution of the differential equation

$$\sum_{\nu=0}^{\tau} \Lambda_\nu(k;x)\; D^{\tau+\nu}P_n(x) = \Xi_{n-\tau}(\tau)P_n(x), \quad n \geqslant 0. \tag{2.2.39}$$

Moreover, with the convention $P_n^{[0]} := P_n$, there is no danger to consider in (2.2.39) the case where $\tau=0$ since it is identically satisfied.

### 2.2.2 Powers of the Bochner's operator

If the elements of a classical sequence are eigenfunctions of Bochner differential operator, shouldn't they also be eigenfunctions of any of its powers?

Even if so, the even order differential operator obtained in theorem 2.2.1 may be represented as a polynomial in the Bochner's operator?

Denoting by $\mathcal{F}^k$ the $k$-th power of the second order differential operator $\mathcal{F}$ given in (2.0.3), we successively define the $k$-th power of $\mathcal{F}$ as $\mathcal{F}^0[y](x) := y(x)$ and $\mathcal{F}^k[y](x) = \mathcal{F}\big(\mathcal{F}^{k-1}[y](x)\big)$, for any $k\in\mathbb{N}^*$ and $y\in\mathcal{P}$.

As a direct consequence of the Bochner's property for the classical polynomial sequences mentioned on page 32, we present the following result.

**Corollary 2.2.4.** *Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical OPS and $k$ a positive integer. Consider the differential operator $\mathcal{F}$ given by (2.0.3) where $\Phi$ represents a monic polynomial with $\deg\Phi \leqslant 2$, and $\Psi$ a polynomial such that $\deg\Psi = 1$. Then, for any set $\{c_{k,\mu}\,:\,0\leqslant\mu\leqslant k\}$ of complex numbers not depending on $n$, each element of $\{P_n\}_{n\in\mathbb{N}}$ fulfils the differential equation given by*

$$\sum_{\mu=0}^{k} c_{k,\mu}\,\mathcal{F}^\mu P_n(x) = \sum_{\mu=0}^{k} c_{k,\mu}\,(\chi_n)^\mu\;P_n(x)\;, \quad n\in\mathbb{N}, \tag{2.2.40}$$

*where $\{\chi_n\}_{n\geqslant 1}$ represents a sequence of nonzero complex numbers.*

*Proof.* Since $\{P_n\}_{n\in\mathbb{N}}$ is a classical OPS, then, according to Bochner's property, there is a monic polynomial $\Phi$ with $\deg\Phi \leqslant 2$, a polynomial $\Psi$ with $\deg\Psi = 1$ and a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ with $\chi_0 = 0$ and $\chi_{n+1} \neq 0$, $n \in \mathbb{N}$, such that (2.0.2) holds. Let us suppose that, for $\nu - 1 \geqslant 1$, $P_n$ is a solution of the differential equation given by $\mathcal{F}^{\nu-1}\, P_n(x) = (\chi_n)^{\nu-1} P_n(x)\,,\ n \in \mathbb{N}$.
Under the assumption we have $\mathcal{F}^{\nu} P_n(x) = \mathcal{F}\left(\mathcal{F}^{\nu-1} P_n(x)\right) = \mathcal{F}\Big((\chi_n)^{\nu-1} P_n(x)\Big)\,.$
On account of (2.0.2) we easily deduce that

$$\mathcal{F}^{\nu} P_n(x) = (\chi_n)^{\nu}\, P_n(x), \quad n \in \mathbb{N},$$

holds for any integer $\nu \geqslant 1$. If $\{c_{k,\mu}\}_{0\leqslant\mu\leqslant k}$ represents any set of complex numbers not depending on $n$, (2.2.40) is trivially verified. □

As a consequence of theorem 2.2.1 and corollary 2.2.4 we present the following result.

**Corollary 2.2.5.** *[75] Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical sequence and $k$ a positive integer. If there exist coefficients $d_{k,\mu}$ and $\widetilde{d}_{k,\mu}$ $0 \leqslant \mu \leqslant k$, not depending on $n$, such that*

$$\Xi_{n-k}(k) = \sum_{\tau=0}^{k} d_{k,\tau}\,(\chi_n)^{\tau}\,, \quad n \geqslant 0, \tag{2.2.41}$$

$$(\chi_n)^k = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau}\, \Xi_{n-\tau}(\tau)\,, \quad n \geqslant 0, \tag{2.2.42}$$

*where $\chi_n$ and $\Xi_{n-\tau}(\tau)$, $1 \leqslant \tau \leqslant k$, $n \geqslant 0$, are respectively the ones presented in (2.0.2) and (2.2.3), then the two following equalities hold:*

$$\sum_{\nu=0}^{k} \Lambda_k(k;x) D^{k+\nu} = \sum_{\tau=0}^{k} d_{k,\tau}\, \mathcal{F}^{\tau}\,, \tag{2.2.43}$$

$$\mathcal{F}^k = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau}\, \left\{\sum_{\nu=0}^{\tau} \Lambda_\nu(\tau;x) D^{\tau+\nu}\right\} \tag{2.2.44}$$

*where $\mathcal{F}$ is given by (2.0.3) and $\left\{\sum_{\nu=0}^{\tau} \Lambda_\nu(\tau;x) D^{\nu+\tau}\right\}$ the one presented in (2.2.39).*

*Proof.* Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical MOPS and $k \geqslant 1$. First we are going to show how (2.2.41) implies (2.2.43) and afterwards how (2.2.42) implies (2.2.44). According to theorem 2.2.1, $P_n$ fulfils the equation

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x) D^{\nu+k} P_n(x) = \Xi_{n-k}(k)\, P_n(x)\,, \quad n \geqslant k.$$

It is clear, from (2.2.3), that whenever $n$ is an integer such that $0 \leqslant n \leqslant k-1$, $\Xi_{n-k}(k) = 0$. So, we actually deduce from theorem 2.2.1,, that

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x) D^{\nu+k} P_n(x) = \Xi_{n-k}(k)\, P_n(x)\,, \quad n \geqslant 0.$$

If $\{d_{k,\tau} \,:\, 0 \leqslant \tau \leqslant k\}$ represents a set of coefficients such that (2.2.41) holds, then we have

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x) D^{\nu+k} P_n(x) = \sum_{\tau=0}^{k} d_{k,\tau}\, (\chi_n)^\tau\, P_n(x)\,, \quad n \geqslant 0,$$

where $\chi_n$ corresponds to the eigenvalues of (2.0.2). On the other hand, corollary 2.2.4 allows us to write

$$\sum_{\mu=0}^{k} d_{k,\mu}\, (\chi_n)^\mu\, P_n(x) = \sum_{\mu=0}^{k} d_{k,\mu}\, \mathcal{F}^\mu P_n(x)\,, \quad n \geqslant 0.$$

Hence we get

$$\mathcal{L}_{2k}\, P_n(x) = 0\,, \quad n \geqslant 0. \tag{2.2.45}$$

where $\mathcal{L}_{2k} = \sum_{\mu=0}^{k} d_{k,\mu}\, \mathcal{F}^\mu - \sum_{\nu=k}^{2k} \Lambda_{\nu-k}(k;x) D^\nu$. Since $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathcal{P}$, then (2.2.45) provides that $\mathcal{L}_{2k} f = 0$, for any $f \in \mathcal{P}$, whence we get (2.2.43).

Likewise, by virtue of corollary 2.2.4 and by taking into account (2.2.42), from (2.2.39) we derive

$$\mathcal{F}^k\, P_n(x) = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau} \left\{ \sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\;\, D^{\tau+\nu} \right\} P_n(x)\,,\ n \in \mathbb{N},$$

which implies the relation (2.2.44), regarding the fact that $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathcal{P}$. □

We intend to know whether it is possible to express the eigenvalues of the differential equation (2.2.1) as a sum of powers of the eigenvalues of the differential equation (2.0.2).

In other words, we face the problem of finding two sets of coefficients $\{d_{k,\tau} \,:\, 1 \leqslant \tau \leqslant k, k \geqslant 1\}$ and $\{\widetilde{d}_{k,\tau} \,:\, 1 \leqslant \tau \leqslant k\,,\ k \geqslant 1\}$ realising the equalities (2.2.41)-(2.2.42). Considering the information contained either in table 2.1 or in table 2.2, we realise that the determination of those two sets of coefficients shall be done separately for each one of the classical families. Indeed, observing the nature of the eigenvalues $\chi_n$ and $\Xi_{n-\tau}(\tau)$, the problem under analysis resembles the relation between the powers of a variable and its factorials. The bridge between those two sequences can be done in a natural way through the Stirling numbers. In order to have a more clear understanding, in the next section we review some basic concepts concerning this subject. That revision suffices to derive the expression for $d_{k,\tau}$ and $\widetilde{d}_{k,\tau}$ (presented in the relations (2.2.41)-(2.2.42)) for the cases of Hermite and Laguerre families, while for the analysis of the cases of Bessel or Jacobi families we introduce a slight modification in the concepts of the factorial of a complex number and *Stirling* numbers.

### 2.2.3 Sums relating a power of a variable and its factorials

The Stirling numbers arise in the search of a bridge between powers of a number and its (shifted) factorials. So, before entering into details, we shall make some considerations about the factorial of a number and the notation that will be in use.

Given a complex number $z$, one may consider its powers $z^n := \prod_{\tau=0}^{n-1} z$, for $n \in \mathbb{N}^*$, and also its shifted factorials, namely its falling factorials $z(z-1)\ldots(z-n+1)$ or its rising factorials $z(z+1)\ldots(z+n-1)$. As far as we are concerned there is no standard notation among mathematicians for either of these factorials. For instance, almost everyone, specially those who work in special functions, use the symbol $(z)_k$ to denote the rising factorial of $z$ and is commonly called as *Pochhammer symbol* . However, some combinatorialists use this same symbol to denote the falling factorial of $z$, among them we quote Louis Comtet [30] or John Riordan [95, 96]. Therefore the reader shall be aware of the notation in use in this text.

The *falling factorial* of a complex number $z$ is denoted by $\{z\}_{(\mathbf{n})}$ and is defined by

$$\{z\}_{(\mathbf{n})} := \begin{cases} 1 & if \quad n=0 \\ \displaystyle\prod_{\tau=0}^{n-1}(z-\tau) & if \quad n \in \mathbb{N}^* \end{cases} \tag{2.2.46}$$

and the *rising factorial*, which is denoted as $(z)_n$ or as $(z)_{n}$ (to maintain coherence with the notation of falling factorial) and is defined by

$$(z)_n := \begin{cases} 1 & if \quad n=0 \\ \displaystyle\prod_{\tau=0}^{n-1}(z+\tau) & if \quad n \in \mathbb{N}^* \ . \end{cases} \tag{2.2.47}$$

Another representation of the falling or rising factorial of a number $z$ can be obtained through the *Gamma* function represented by $\Gamma(\cdot)$ and defined by $\Gamma(z) = \int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\mathrm{dt}$ when $\Re(z) > 0$, and $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0$ and $\Gamma(1) = 1$. From the definition of falling and rising factorials, it follows:

$$\{z\}_{(\mathbf{n})} = \frac{\Gamma(z+1)}{\Gamma(z-n+1)} \qquad ; \qquad (z)_n := \frac{\Gamma(z+n)}{\Gamma(z)}$$

As a direct consequence of the definition, for any $z \in \mathbb{C}$ and $n \in \mathbb{N}$, the following identities hold:

$$\{z\}_{(\mathbf{n})} = (-1)^n \, (-z)_n$$
$$(z)_n = (-1)^n \, \{-z\}_{(\mathbf{n})}$$
$$\{z+n\}_{(\mathbf{n})} = (z+1)_n$$

Representing by $s(k,\nu)$ and $S(k,\nu)$, with $k,\nu \in \mathbb{N}$, the **Stirling numbers of first and second kind**, respectively, the following equalities hold [30, 95, 96]:

$$\{x\}_{(\mathbf{k})} = \sum_{\nu=0}^{k} s(k,\nu)\, x^{\nu} \, . \tag{2.2.48}$$

and

$$x^k = \sum_{\nu=0}^{k} S(k,\nu)\, \{x\}_{(\nu)} \, , \tag{2.2.49}$$

where $\{x\}_{(\mathbf{k})}$ represent the **falling factorial** of $x$ and is defined in (**??**). Such numbers fulfil a "*triangular*" recurrence relation; Namely we have

$$\begin{cases} s(k+1,\nu+1) = s(k,\nu) - k\, s(k,\nu+1) \\ s(k,0) = s(0,k) = \delta_{k,0} \\ s(k,\nu) = 0\,, \quad \nu \geqslant k+1 \end{cases}$$

and

$$\begin{cases} S(k+1,\nu+1) = S(k,\nu) + (\nu+1)\, S(k,\nu+1) \\ S(k,0) = S(0,k) = \delta_{k,0} \\ S(k,\nu) = 0\,, \quad \nu \geqslant k+1 \end{cases}$$

with $k,\nu \in \mathbb{N}$ (see, for instance the book of L. Comtet [30, Chapter V]). The Stirling numbers of first and second kind fulfil the biorthogonality conditions

$$\sum_{\tau=0}^{\max\{k,\nu\}} s(k,\tau)\, S(\tau,\nu) = \sum_{\tau=0}^{\max\{k,\nu\}} S(k,\tau)\, s(\tau,\nu) = \delta_{k,\nu} \, .$$

The matrix $\mathbf{s} := [s(k,\nu)]_{k,\nu\in\mathbb{N}}$ consisting of the Stirling numbers of the first kind is the inverse of the matrix $\mathbf{S} := [S(k,\nu)]_{k,\nu\in\mathbb{N}}$ of the Stirling numbers of the second kind ($\mathbf{s}^{-1} = \mathbf{S}$). It is also possible to represent the Stirling numbers of the second kind $S(k,\nu)$ in a closed form:

$$S(k,\nu) = \frac{1}{\nu!} \sum_{\tau=0}^{\nu} (-1)^{\nu-\tau} \binom{\nu}{\tau} \tau^k \, , \quad 1 \leqslant \nu \leqslant k \, .$$

We shall make once more some considerations about the notation in use, this time towards the Stirling numbers, which apparently has never been standardised. In the chapter 24 of the book of Abramowitz and Stegun [2] are resumed some of the notations used for Stirling numbers. Despite their recommendation, we will follow the one suggested by Comtet [30] or Riordan [96].

We now introduce a slight modification on the concept of the falling factorial of a number.

**Definition 2.2.6.** Let $A$ be a number (possibly complex) and $k \in \mathbb{N}$. For any number $x$ we define

$$\{x\}_{(\mathbf{k};\mathbf{A})} := \begin{cases} 1 & if \quad k = 0, \\ \displaystyle\prod_{\nu=0}^{k-1} (x - \nu(\nu + A)) & if \quad k \in \mathbb{N}^*, \end{cases} \tag{2.2.50}$$

to be the **$\mathbf{A}$-modified falling factorial (of order $k$)**.

It is clear that both $\left\{\{x\}_{(\mathbf{n};\mathbf{A})}\right\}_{n\in\mathbb{N}}$ and $\{x^n\}_{n\in\mathbb{N}}$ are a set of independent polynomials spanning $\mathcal{P}$ (in brief, are two MPS). As a result, there exist two unique sequences of numbers $\{\widehat{s}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$ and $\{\widehat{S}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$ such that

$$\{x\}_{(\mathbf{k};\mathbf{A})} = \sum_{\nu=0}^{k} \widehat{s}_A(k,\nu)\, x^\nu \ , \ k \in \mathbb{N} \tag{2.2.51}$$

$$x^k = \sum_{\nu=0}^{k} \widehat{S}_A(k,\nu)\, \{x\}_{(\nu;\mathbf{A})} \ , \ k \in \mathbb{N}, \tag{2.2.52}$$

Now, the issue in hand is to find information about these two number sequences $\{\widehat{s}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$ and $\{\widehat{S}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$. To accomplish this goal we present the next result.

**Proposition 2.2.7.** *[75] The numbers $\widehat{s}_A(k,\nu)$ defined by (2.2.51) satisfy the following "triangular" recurrence relation*

$$\widehat{s}_A(k+1,\nu+1) = \widehat{s}_A(k,\nu) - k(k+A)\,\widehat{s}_A(k,\nu+1) \ , \tag{2.2.53}$$

$$\widehat{s}_A(k,0) = \widehat{s}_A(0,k) = \delta_{k,0} \ , \tag{2.2.54}$$

$$\widehat{s}_A(k,\nu) = 0 \, , \nu \geqslant k+1 \ , \tag{2.2.55}$$

*whereas $\widehat{S}_A(k,\nu)$ defined by (2.2.52) satisfy the "triangular" recurrence relation given by*

$$\widehat{S}_A(k+1,\nu+1) = \widehat{S}_A(k,\nu) + (\nu+1)(\nu+1+A)\widehat{S}_A(k,\nu+1) \ , \tag{2.2.56}$$

$$\widehat{S}_A(k,0) = \widehat{S}_A(0,k) = \delta_{k,0} \ , \tag{2.2.57}$$

$$\widehat{S}_A(k,\nu) = 0 \, , \nu \geqslant k+1 \ , \tag{2.2.58}$$

*for $k, \nu \in \mathbb{N}$.*

*Proof.* Suppose that the relations (2.2.51)-(2.2.52) hold. The fact that $x^0 = 1 = \{x\}_{(\mathbf{0};\mathbf{A})}$ provides that $\widehat{s}_A(0,0) = \widehat{S}_A(0,0) = 1$. It is clear that $\{x\}_{(\mathbf{k};\mathbf{A})}$, with $k \in \mathbb{N}$, is a polynomial in $x$ and $\deg\Big(\{x\}_{(\mathbf{k};\mathbf{A})}\Big) = k$. Therefore, the relations (2.2.55) and (2.2.58) are just a consequence of (2.2.51) and (2.2.52), respectively. Meanwhile, due to (2.2.50), the following identity

$$\{x\}_{(\mathbf{k+1};\mathbf{A})} = \big(x - k(k+A)\big)\,\{x\}_{(\mathbf{k};\mathbf{A})}\ ,\ k \in \mathbb{N}, \tag{2.2.59}$$

holds. Therefore, we successively have:

$$\begin{aligned}
\sum_{\nu=0}^{k+1} \widehat{s}_A(k+1,\nu)x^\nu &= \{x\}_{(\mathbf{k+1};\mathbf{A})} = \big(x - k(k+A)\big)\,\{x\}_{(\mathbf{k};\mathbf{A})} \\
&= \big(x - k(k+A)\big) \sum_{\nu=0}^{k} \widehat{s}_A(k,\nu)\,x^\nu \\
&= \sum_{\nu=1}^{k} \big\{\widehat{s}_A(k,\nu-1) - k(k+A)\widehat{s}_A(k,\nu)\big\}x^\nu \\
&\quad + \widehat{s}_A(k,k)\,x^{k+1} - k(k+A)\widehat{s}_A(k,0), \quad k \in \mathbb{N}.
\end{aligned}$$

Equating the coefficients of $x^\nu$, $0 \leqslant \nu \leqslant k$, in the first and last members of the previous equalities permits to deduce

$$\widehat{s}_A(k+1,0) = -k(k+A)\widehat{s}_A(k,0) \quad , \quad \widehat{s}_A(k+1,k+1) = \widehat{s}_A(k,k) \tag{2.2.60}$$

and also (2.2.53) with $\nu$ replaced by $\nu+1$. Clearly, (2.2.60) implies (2.2.54).

Likewise, from (2.2.52), we deduce

$$\sum_{\nu=0}^{k+1} \widehat{S}_A(k+1,\nu)\{x\}_{(\nu;\mathbf{A})} = x^{k+1} = x \cdot x^k = \sum_{\nu=0}^{k+1} \widehat{S}_A(k,\nu)\ x\ \{x\}_{(\nu;\mathbf{A})}$$

which, on account of (2.2.59) with $k$ replaced by $\nu$, becomes

$$\sum_{\nu=0}^{k+1} \widehat{S}_A(k+1,\nu)\{x\}_{(\nu;\mathbf{A})} = \sum_{\nu=0}^{k} \widehat{S}_A(k,\nu)\big\{\{x\}_{(\nu+\mathbf{1};\mathbf{A})} + \nu(\nu+A)\{x\}_{(\nu;\mathbf{A})}\big\}$$

As $\{\{x\}_{(\nu;\mathbf{A})}\}_{\nu\in\mathbb{N}}$ forms an independent system on $\mathcal{P}$, we conclude that $\widehat{S}_A(k+1,0) = 0$, $\widehat{S}_A(k+1,k+1) = \widehat{S}_A(k,k)$ and also (2.2.56) after replacing $\nu$ by $\nu+1$. Thus, we have (2.2.57). $\square$

The insertion of (2.2.51) into (2.2.52) brings $x^k = \sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu} \widehat{S}_A(k,\nu)\ \widehat{s}_A(\nu,\tau)x^\tau$ , and it yields

$$\sum_{\nu\in\mathbb{N}} \widehat{S}_A(k,\nu)\ \widehat{s}_A(\nu,\tau) = \delta_{k,\tau}\ .$$

Conversely, if we insert (2.2.52) into (2.2.51), we deduce

$$\sum_{\nu\in\mathbb{N}} \widehat{s}_A(k,\nu)\, \widehat{S}_A(\nu,\tau) = \delta_{k,\tau}\,.$$

The similar-look of $\widehat{s}_A(k,\nu)$ and $\widehat{S}_A(k,\nu)$ with the Stirling numbers of first and second kind, respectively, compels us to call the numbers $\widehat{s}_A(k,\nu)$ and $\widehat{S}_A(k,\nu)$ as the **A-modified Stirling numbers of first and second kind**, respectively. Several authors have studied the Stirling numbers, its generalisations or some of their analogies (among them we quote Chou et al. [28], Hsu and Shiue [56], Milne and Bhatnagar [92]), however, as far as we are concerned, the study of $\widehat{s}_A(k,\nu)$ and $\widehat{S}_A(k,\nu)$ still remains somewhat unexplored, except for some particular values of $A$ (this will be explained in due time). It might be worthy to explore other properties about the so called $A$-modified Stirling numbers. Either way, this is not the issue for the moment, so we will leave the study of other potential interesting properties for a future work. Nevertheless, we present some few considerations, specially those about the $A$-modified Stirling numbers of the second kind.

**Corollary 2.2.8.** *[75] The numbers $\widehat{S}_A(k,\nu)$ presented in (2.2.52) equal*

$$\widehat{S}_A(k,\nu) = \frac{1}{\nu!}\sum_{\sigma=1}^{\nu}\binom{\nu}{\sigma}(-1)^{\nu+\sigma}\,\frac{(A+2\,\sigma)\,\Gamma(A+\sigma)}{\Gamma(A+\sigma+\nu+1)}\,\Big(\sigma(\sigma+A)\Big)^k, \tag{2.2.61}$$

*for $1\leqslant\nu\leqslant k$.*

*Proof.* From proposition 2.2.7, it follows that (2.2.52) holds for all the integers $k\in\mathbb{N}$ where the numbers $\widehat{S}_A(k,\nu)$ satisfy the relations (2.2.56)-(2.2.58). Now, let

$$c_{k,\nu}(A) = \frac{1}{\nu!}\sum_{\sigma=1}^{\nu}\binom{\nu}{\sigma}(-1)^{\nu+\sigma}\,\frac{(A+2\,\sigma)\,\Gamma(A+\sigma)}{\Gamma(A+\sigma+\nu+1)}\,\Big(\sigma(\sigma+A)\Big)^k,\ 1\leqslant\nu\leqslant k\,.$$

When we take $\nu=0$ in (2.2.56), we get

$$\widehat{S}_A(k+1,1) = \begin{cases} 1 & ,\ k=0 \\ (A+1)\,\widehat{S}_A(k,1) & ,\ k\geqslant 1 \end{cases},$$

therefore

$$\widehat{S}_A(k,1) = (A+1)^{k-1}\,,\ k\geqslant 1. \tag{2.2.62}$$

Now, the relation (2.2.56) with $\nu=1$ and on account of (2.2.62) becomes

$$\widehat{S}_A(k+1,2) = (A+1)^{n-1} + 2(A+2)\widehat{S}_A(k,2),\ k\geqslant 2,$$

from which we derive

$$\begin{aligned}\widehat{S}_A(k,2) &= \frac{\big(2(2+A)\big)^{n-1} - 2(1+A)^{n-1}}{2(3+A)}\\ &= \frac{1}{2}\left\{\frac{\big(2(2+A)\big)^{k}\,(A+4)\,\Gamma(A+2)}{\Gamma(A+5)} - 2\frac{(1+A)^k\,(A+2)\,\Gamma(A+1)}{\Gamma(A+4)}\right\}\end{aligned} \tag{2.2.63}$$

for all the integers $k \geqslant 2$. Hence (2.2.62)-(2.2.63) show that $\widehat{S}_A(k,\nu) = c_{k,\nu}(A)$ for $\nu = 1, 2$ and $k \geqslant 1$.
Now suppose that $\widehat{S}_A(k,\nu) = c_{k,\nu}(A)$ for $1 \leqslant \nu \leqslant k$. From (2.2.56), we have

$$\widehat{S}_A(k+1,\nu) = \widehat{S}_A(k,\nu-1) + \big(\nu(\nu+A)\big)\widehat{S}_A(k,\nu)$$

$$= c_{k,\nu-1}(A) + \big(\nu(\nu+A)\big)\; c_{k,\nu}(A)$$

$$= \frac{\nu(\nu+A)\,(A+2\nu)\,\Gamma(A+\nu)}{\nu!\;\Gamma(A+2\nu+1)}\;\big(\nu(\nu+A)\big)^k$$

$$+\sum_{\sigma=1}^{\nu-1}\left\{-1+\frac{\nu(\nu+A)}{(\nu-\sigma)(A+\sigma+\nu)}\right\}\frac{(-1)^{\nu+\sigma}\,(A+2\sigma)\,\Gamma(A+\sigma)\,\big(\sigma(\sigma+A)\big)^k}{(\nu-\sigma-1)!\;\sigma!\;\Gamma(A+\sigma+\nu)}$$

$$= \frac{\nu(\nu+A)\,(A+2\nu)\,\Gamma(A+\nu)}{\nu!\;\Gamma(A+2\nu+1)}\;\big(\nu(\nu+A)\big)^k$$

$$+\sum_{\sigma=1}^{\nu-1}\frac{\sigma(\sigma+A)\;(-1)^{\nu+\sigma}\;(A+2\sigma)\;\Gamma(A+\sigma)\;\big(\sigma(\sigma+A)\big)^k}{(\nu-\sigma)!\;\sigma!\;\Gamma(A+\sigma+\nu+1)}$$

$$= c_{k+1,\nu}(A)\,, \quad 1 \leqslant \nu \leqslant k+1,$$

whence we conclude that $\widehat{S}_A(k,\nu) = c_{k,\nu}(A)$ for all $k,\nu \in \mathbb{N}^*$ with $\nu \leqslant k$. □

**Remark 2.2.3.** When $x = n(n+A)$ for $n \in \mathbb{N}$ and $A \in \mathbb{C}$, its $A$-modified factorial (of order $k$) is given by:

$$\{n(n+A)\}_{(\mathbf{k};\mathbf{A})} = \prod_{\nu=0}^{k-1}\Big(n(n+A) - \nu(\nu+A)\Big) = \prod_{\nu=0}^{k-1}\Big((n-\nu)(n+A+\nu)\Big)$$

which, in accordance with (**??**)-(**??**), may be expressed like

$$\{n(n+A)\}_{(\mathbf{k};\mathbf{A})} = \{n\}_{(\mathbf{k})}\;(n+A)_k = \{n\}_{(\mathbf{k})}\;(n+A+k-1)_k\,. \tag{2.2.64}$$

The previous equalities highlight a relation between the $A$-modified Stirling numbers and the Stirling numbers itself. Namely, recalling (2.2.51) and (2.2.48), the comparison of the first and last members of the previous equality, may be transformed into

$$\sum_{\nu=0}^{k} \widehat{s}_A(k,\nu)\ \big(n(n+A)\big)^{\nu} = \sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu} s(k,\nu)\ s(\nu,\tau)\ n^{\nu}\ (n+A+k-1)^{\tau}$$

or, equivalently,

$$\sum_{\nu=0}^{k} \widehat{s}_A(k,\nu)\ \big(n(n+A)\big)^{\nu} = \sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu} (-1)^{\nu+\tau}\ s(k,\nu)\ s(\nu,\tau)\ n^{\nu}\ (n+A)^{\tau}$$

Such expression may be simplified, nevertheless, once again, we will leave the study of the properties of such numbers to a future work because we need to delimit the study. Analogously, due to (2.2.52) and (2.2.49), from the relation $\big(n(n+A)\big)^{k} = n^{k}\ (n+A)^{k}$ we derive

$$\sum_{\nu=0}^{k} \widehat{S}_A(k,\nu)\ \{n(n+A)\}_{(\nu;\mathbf{A})} = \sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu} S(k,\nu)\ S(\nu,\tau)\ \{n\}_{(\nu)}\ \{n+A\}_{(\tau)}\ .$$

In Tables 2.3 (p.66) and 2.4 (p.67) we present the first computed $A$-modified Stirling numbers of first and second kind, respectively.

### 2.2.4 Sums relating powers of Bochner differential operator and the obtained even order differential operator

This section aims to explicitly present the $2k$-order differential equation (2.2.1) given in theorem 2.2.1, for each classical family (Hermite, Laguerre, Bessel and Jacobi) and any integer $k \geqslant 1$. The expression for the polynomials $\Lambda_{\nu}(k;\cdot)$ (with $0 \leqslant \nu \leqslant k$) that will be in use is the one given in theorem 2.2.3, in spite of the one given by (2.2.2).

Following corollary 2.2.5 it is possible to express the even order differential operator associated to the equation (2.2.1) as a polynomial in $\mathcal{F}$, the Bochner differential operator, providing there is a set of numbers $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ such that the condition (2.2.41) holds true. Conversely, if there is a set of numbers $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ such that (2.2.42) holds, then we obtain an explicit expression for any power of the Bochner's operator according to (2.2.44) and considering (2.2.29).

The determination of the sets $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ and $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ will be thoroughly revealed for each classical family, by taking into account the considerations made in section 2.2.3. To accomplish this issue, we will work separately with each one of the classical families. Naturally, it won't be necessary to compute the successive powers of the Bochner's operator $\mathcal{F}$. For the sequel we will strongly use the information contained in Table 2.1 and Table 2.2.

**Hermite case**

Let $\{P_n(\cdot)\}_{n\in\mathbb{N}}$ be an Hermite monic polynomial sequence. Based on the information given in Table 2.2 and according to (2.2.3)-(2.2.4) we get $\Xi_n(k) = (-2)^k \, \{n+k\}_{(\mathbf{k})}$ , $n \in \mathbb{N}$. On the other hand, considering the information provided by Table 2.1, the coefficients defined in (2.2.30) are simply like $\omega_{k,\nu} = (-2)^{-\nu}$ , $0 \leqslant \nu \leqslant k$ . Therefore, the polynomial $\Lambda_\nu(k;x)$ defined in (2.2.29) may be expressed as follows:

$$\Lambda_\nu(k;x) = \frac{1}{\nu!} \, (-2)^{k-\nu} \, (P_k)^{(\nu)} = \binom{k}{\nu} \, (-2)^{k-\nu} \, P^{[\nu]}_{k-\nu} \, , \quad 0 \leqslant \nu \leqslant k.$$

Following (2.1.3), for each integer $\nu \geqslant 1$, $P^{[\nu]}_n(\cdot) = P_n(\cdot)$, $n \in \mathbb{N}$, therefore

$$\Lambda_\nu(k;x) = \binom{k}{\nu} \, (-2)^{k-\nu} \, P_{k-\nu}(x) \, , \; 0 \leqslant \nu \leqslant k, \tag{2.2.65}$$

where

$$P_{2\tau}(x) = (2\tau)! \sum_{\mu=0}^{\tau} \frac{(-1)^{\tau-\mu}}{2^{2(\tau-\mu)} \, (\tau-\mu)!} \, \frac{x^{2\mu}}{(2\mu)!} \, , \quad \tau \in \mathbb{N} \, ,$$
$$P_{2\tau+1}(x) = (2\tau+1)! \sum_{\mu=0}^{\tau} \frac{(-1)^{\tau-\mu}}{2^{2(\tau-\mu)} \, (\tau-\mu)!} \, \frac{x^{2\mu+1}}{(2\mu+1)!} \, , \quad \tau \in \mathbb{N} \, .$$

Thus, $Y(x) = P_n(x)$ is a solution of the following differential equation:

$$\boxed{\sum_{\nu=0}^{k} \binom{k}{\nu} \, (-2)^{-\nu} \, P_{k-\nu}(x) \;\; D^{k+\nu} Y(x) = \{n\}_{(\mathbf{k})} \; Y(x) \, , \quad n \in \mathbb{N} \, .}$$

The relation (2.2.48) with $x$ replaced by $n$ allows to deduce a sum relating $\Xi_{n-k}(k)$ and $\chi_n$ given in Table 2.1 and it goes as follows:

$$\Xi_{n-k}(k) = (-2)^k \{n\}_{(\mathbf{k})} \; = (-2)^k \, \sum_{\tau=0}^{k} s(k,\tau) \, n^\tau = \sum_{\tau=0}^{k} (-2)^{k-\tau} \, s(k,\tau) \, \big(\chi_n\big)^\tau, \; n \in \mathbb{N},$$

where $s(k,\tau)$, with $0 \leqslant \tau \leqslant k$, represent the Stirling numbers of first kind defined in (2.2.48). The first and last members of the previous equalities correspond to (2.2.41) with

$$d_{k,\tau} = (-2)^{k-\tau} \, s(k,\tau) \, , \quad 0 \leqslant \tau \leqslant k \, .$$

Conversely, on account of (2.2.49) with $x$ replaced by $n$, we derive

$$(\chi_n)^k = (-2)^k \sum_{\tau=0}^{k} S(k,\tau) \{n\}_{(\tau)} \; = \sum_{\tau=0}^{k} (-2)^{k-\tau} \, S(k,\tau) \, \Xi_{n-\tau}(\tau), \quad n \in \mathbb{N},$$

where $S(k,\tau)$, with $0 \leqslant \tau \leqslant k$, represent the Stirling numbers of second kind. Thus, we have just obtained (2.2.42) if we consider

$$\widetilde{d}_{k,\tau} = (-2)^{k-\tau}\, S(k,\tau)\,, \quad 0 \leqslant \tau \leqslant k\,.$$

As a result, by virtue of corollary 2.2.5, we conclude

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\, D^{k+\nu} = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, s(k,\tau)\, \mathcal{F}^{\tau} \\ \displaystyle\mathcal{F}^{k} = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, S(k,\tau) \sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\, D^{\tau+\nu} \end{cases}, \tag{2.2.66}$$

where $\Lambda_\nu(k;x)$ is given in (2.2.65) and, considering Table 2.1, $\mathcal{F} = D^2 - 2x\,D$.

### Laguerre case

Consider $\{P_n(\cdot;\alpha)\}_{n\in\mathbb{N}}$ with $\alpha \neq -(n+1)$, $n \in \mathbb{N}$, to be a Laguerre monic polynomial sequence. The information contained in Table 2.2 enables $\lambda_n^k = \dfrac{(-1)^k}{(\alpha+1)_k} = \lambda_0^k$ and also for $\Xi_n(k) = \dfrac{(-1)^k}{(\alpha+1)_k}\,\{n+k\}_{(\mathbf{k})}\,,\ n \in \mathbb{N}$ in accordance with (2.2.3). Following the information of Table 2.1 for the Laguerre case, according to (2.2.30) we have $\omega_{k,\nu} = (-1)^{-\nu}$, (with $0 \leqslant \nu \leqslant k$ ) and the polynomial $\Lambda_\nu(k;x)$ defined in (2.2.29) may be expressed as follows:

$$\Lambda_\nu(k;x) = \frac{1}{\nu!}\,\frac{(-1)^{k-\nu}}{(\alpha+1)_k}\, x^\nu\, (P_k)^{(\nu)} = \binom{k}{\nu}\frac{(-1)^{k-\nu}}{(\alpha+1)_k}\, x^\nu\, P_{k-\nu}^{[\nu]}(x;\alpha)$$

Since, in accordance with (2.1.3), for each integer $\nu \geqslant 1$, $P_n^{[\nu]}(\cdot;\alpha) = P_n(\cdot,\alpha+\nu)$, $n \in \mathbb{N}$, then we have

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\frac{(-1)^{k-\nu}}{(\alpha+1)_k}\, x^\nu\, P_{k-\nu}(x;\alpha+\nu) \tag{2.2.67}$$

with

$$P_{k-\nu}(x;\alpha+\nu) = \sum_{\mu=0}^{k-\nu} \binom{k-\nu}{\mu} (-1)^{k-\nu-\mu}\, \frac{\Gamma(k+\alpha+1)}{\Gamma(\mu+\alpha+\nu+1)}\, x^\mu\,, \quad 0 \leqslant \nu \leqslant k\,.$$

Following (2.2.1), $Y(x) = P_n(x;\alpha)$ is a solution of the differential equation

$$\boxed{\sum_{\nu=0}^{k} \binom{k}{\nu} \Big\{ (-1)^\nu\, x^\nu\, P_{k-\nu}(x;\alpha+\nu) \Big\}\, D^{k+\nu}\big(Y(x)\big) = \{n\}_{(\mathbf{k})}\, Y(x)\,, \quad n \in \mathbb{N}\,.}$$

The problem of determining the two sets of coefficients $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ and $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ realising the conditions (2.2.41)-(2.2.42) in this case, is analogous to the corresponding problem in the Hermite case. Indeed, if we replace $x$ by $n$ in (2.2.48), then the eigenvalues $\Xi_{n-k}(k)$ become:

$$\Xi_{n-k}(k) = \frac{(-1)^k}{(\alpha+1)_k} \sum_{\nu=0}^{k} s(k,\nu)\, n^\nu = \sum_{\nu=0}^{k} \frac{(-1)^k}{(\alpha+k)_k}\, s(k,\nu)\, \big(\chi_n\big)^\nu\,, \quad n \in \mathbb{N},$$

providing (2.2.41) with

$$d_{k,\tau} = \frac{(-1)^{k-\tau}}{(\alpha+1)_k}\, s(k,\tau)\,, \quad 0 \leqslant \tau \leqslant k\,.$$

Conversely, we have

$$\begin{aligned} (\chi_n)^k &= (-1)^k\, n^k = (-1)^k \sum_{\tau=0}^{k} S(k,\tau)\, \{n\}_{(\tau)} \\ &= \sum_{\tau=0}^{k} (-1)^k\, S(k,\tau) \left(\frac{(-1)^\tau}{(\alpha+1)_\tau}\right)^{-1} \Xi_{n-\tau}(\tau)\,, \quad n \in \mathbb{N}, \end{aligned}$$

whence we attain (2.2.42) with

$$\widetilde{d}_{k,\tau} = (-1)^{k-\tau}\, (\alpha+1)_\tau\, S(k,\tau)\,, \quad 0 \leqslant \tau \leqslant k\,.$$

From corollary 2.2.5 it follows

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\, D^{k+\nu} = \sum_{\tau=0}^{k} \frac{(-1)^{k-\tau}}{(\alpha+1)_k}\, s(k,\tau)\, \mathcal{F}^\tau \\ \displaystyle\mathcal{F}^k = \sum_{\tau=0}^{k} (-1)^{k-\tau}\, (\alpha+1)_\tau\, S(k,\tau) \sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\, D^{\tau+\nu} \end{cases}, \tag{2.2.68}$$

where $\Lambda_\nu(k;x)$ is given by (2.2.67) and, following Table 2.1 and the definition of $\mathcal{F}$ described in (2.0.3), $\mathcal{F} = xD^2 - (x-\alpha-1)D$.

### Bessel case

Let $\{P_n(\cdot;\alpha)\}_{n\in\mathbb{N}}$ with $\alpha \neq -\frac{n}{2}$, $n \in \mathbb{N}$, represent a Bessel monic polynomial sequence. The information given in Table 2.2 permits to obtain $\Xi_n(k) = \lambda_n^k\, \{n+k\}_{(\mathbf{k})}$ with $\lambda_n^k = C_\alpha^k\, \big(2\alpha-1+k+n\big)_k$, for $n \in \mathbb{N}$, where $C_\alpha^k = 4^{-k}\, \big(2\alpha\big)_{2k}$ , in accordance with (2.2.3)-(2.2.4) . Considering the information presented in Table 2.1 for the Bessel case, (2.2.30) becomes $\omega_{k,\nu} = \dfrac{1}{\left(2\alpha+k-1\right)_\nu}$ , (with $0 \leqslant \nu \leqslant k$ ). Following (**??**)-(**??**), we derive that $\Lambda_\nu(k;x)$ , defined in (2.2.29), may be expressed as follows:

$$\Lambda_\nu(k;x) = \binom{k}{\nu} \; C^k_\alpha \; (2\alpha-1+k+\nu)_{k-\nu} \;\; x^{2\nu} \; P^{[\nu]}_{k-\nu}(x;\alpha) \, , \quad 0 \leqslant \nu \leqslant k.$$

By recalling (2.1.3), for each integer $\nu \geqslant 1$, $P^{[\nu]}_n(\cdot;\alpha) = P_n(\cdot,\alpha+\nu)$, $n \in \mathbb{N}$, so, we have

$$\Lambda_\nu(k;x) = \binom{k}{\nu} \; C^k_\alpha \; (2\alpha-1+k+\nu)_{k-\nu} \;\; x^{2\nu} \; P_{k-\nu}(x;\alpha+\nu), \; 0 \leqslant \nu \leqslant k, \qquad (2.2.69)$$

where

$$P_{k-\nu}(x;\alpha+\nu) = \sum_{\mu=0}^{k-\nu} \binom{k-\nu}{\mu} \frac{2^{k-\nu-\mu} \;\; x^\mu}{(2\alpha-1+k+\nu+\mu)_{k-\nu-\mu}} \, , \quad 0 \leqslant \nu \leqslant k \, .$$

Following (2.2.1), $Y(x) = P_n(x;\alpha)$ is a solution of the differential equation

$$\boxed{\begin{aligned} &\sum_{\nu=0}^{k} \binom{k}{\nu} \; \big\{(2\alpha-1+k+\nu)_{k-\nu} \, x^{2\nu} \; P_{k-\nu}(x;\alpha+\nu) \; \big\} \;\; D^{k+\nu}\big(Y(x)\big) \\ &= \{n\}_{(\mathbf{k})} \; (2\alpha-1+n)_k \; Y(x) \, , \quad n \in \mathbb{N} \, . \end{aligned}}$$

Now we face the problem of determining the two sets of coefficients $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ and $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ realising the conditions (2.2.41)-(2.2.42) for this case. Indeed, considering (2.2.64) presented in remark 2.2.3 with $A = 2\alpha-1$, we get

$$\Xi_{n-k}(k) = C^k_\alpha \; \{n(n+2\alpha-1)\}_{(\mathbf{k};\mathbf{2}\alpha-\mathbf{1})}$$

and, on account of (2.2.51), we deduce

$$\begin{aligned} \Xi_{n-k}(k) &= C^k_\alpha \sum_{\nu=0}^{k} \widehat{s}_{2\alpha-1}(k,\nu) \; \big(n(n+2\alpha-1)\big)^\nu \\ &= C^k_\alpha \sum_{\nu=0}^{k} \widehat{s}_{2\alpha-1}(k,\nu) \; \big(\chi_n\big)^\nu, \; n \in \mathbb{N}, \end{aligned}$$

according to the expression of $\chi_n$, $n \in \mathbb{N}$, given in Table 2.1. Equating the first and last members of the previous equalities, we obtain (2.2.41) with

$$d_{k,\tau} = C^k_\alpha \; \widehat{s}_{2\alpha-1}(k,\tau) \, , \quad 0 \leqslant \tau \leqslant k.$$

Conversely, by virtue of (2.2.52) we have

$$\begin{aligned} (\chi_n)^k &= \big(n\,(n+2\alpha-1)\big)^k = \sum_{\tau=0}^{k} \widehat{S}_{2\alpha-1}(k,\tau) \; \{n\,(n+2\alpha-1\}_{(\tau;\mathbf{2}\alpha-\mathbf{1})} \\ &= \sum_{\tau=0}^{k} \big(C(\tau;\alpha)\big)^{-1} \, \widehat{S}_{2\alpha-1}(k,\tau) \; \Xi_{n-\tau}(\tau) \, , \quad n \in \mathbb{N}, \end{aligned}$$

whence we get the relation (2.2.42) with

$$\widetilde{d}_{k,\tau} = C(\tau;\alpha)^{-1}\ \widehat{S}_{2\alpha-1}(k,\tau)\,,\quad 0\leqslant\tau\leqslant k.$$

From corollary 2.2.5 it follows

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k}\Lambda_\nu\left(k;x\right)D^{k+\nu} = \sum_{\tau=0}^{k} C_\alpha^k\ \widehat{s}_{2\alpha-1}(k,\nu)\ \mathcal{F}^\tau \\ \displaystyle\mathcal{F}^k = \sum_{\tau=0}^{k}\big(C(\tau;\alpha)\big)^{-1}\ \widehat{S}_{2\alpha-1}(k,\tau)\ \sum_{\nu=0}^{\tau}\Lambda_\nu\left(\tau;x\right)D^{\tau+\nu}\end{cases}, \tag{2.2.70}$$

where $\Lambda_\nu(k;x)$ is given in (2.2.69) and $\mathcal{F} = x^2D^2 + 2(\alpha x+1)D$.

## Jacobi case

Let $\{P_n(\cdot;\alpha,\beta)\}_{n\in\mathbb{N}}$ with $\alpha,\beta\neq-(n+1),\ \alpha+\beta\neq-(n+2),\ n\in\mathbb{N}$, represent a Jacobi monic polynomial sequence. According to (2.2.3) we have $\Xi_n(k)=\lambda_n^k\ \{n+k\}_{(\mathbf{k})}\,,\ n\in\mathbb{N}$, and, the information of Table 2.2 provides $\lambda_n^k = C_{\alpha,\beta}^k\ (\alpha+\beta+1+k+n)_k$ where $C_{\alpha,\beta}^k = \frac{(-4)^{-k}\ (\alpha+\beta+2)_{2k}}{(\alpha+1)_k\ (\beta+1)_k}$ . Now, based on Table 2.1, the formula (2.2.30) becomes $\omega_{k,\nu} = \dfrac{1}{(\alpha+\beta+1+k)_\nu}$ , (with $0\leqslant\nu\leqslant k$ ). On account of (**??**)-(**??**), the polynomial $\Lambda_\nu(k;x)$ , defined in (2.2.29), may be expressed like:

$$\Lambda_\nu(k;x) \quad = \quad \binom{k}{\nu}\ C_{\alpha,\beta}^k\ (\alpha+\beta+1+k+\nu)_{k-\nu}\ (x^2-1)^\nu\ P_{k-\nu}^{[\nu]}(x;\alpha,\beta)$$

Considering (2.1.3), for each integer $\nu\geqslant1$, $P_n^{[\nu]}(\cdot;\alpha,\beta) = P_n(\cdot,\alpha+\nu,\beta+\nu)$, $n\in\mathbb{N}$, whence it follows that

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\ C(k;\alpha,\beta)\ (\alpha+\beta+1+k+\nu)_{k-\nu}\ (x^2-1)^\nu\ P_{k-\nu}(x;\alpha+\nu,\beta+\nu) \tag{2.2.71}$$

where

$$\begin{aligned}P_{k-\nu}(x;\alpha+\nu,\beta+\nu) = \quad & \frac{(-2)^{k-\nu}\Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+1)}\sum_{\mu=0}^{k-\nu}\left\{\sum_{\tau=\mu}^{k-\nu}(-2)^{-\tau}\binom{k-\nu}{\tau}\binom{\tau}{\mu}\right.\\ & \left.\times\frac{\Gamma(\tau+k+\nu+\alpha+\beta+1)}{\Gamma(\tau+\alpha+\nu+1)}\right\}x^\mu\,,\quad 0\leqslant\nu\leqslant k.\end{aligned}$$

Following (2.2.1), $Y(x) = P_n(x;\alpha,\beta)$ is a solution of the following differential equation

$$\boxed{\begin{aligned}&\sum_{\nu=0}^{k}\binom{k}{\nu}\ \big\{(\alpha+\beta+1+k+\nu)_{k-\nu}\ (x^2-1)^{\nu}\ P_{k-\nu}(x;\alpha+\nu,\beta+\nu)\ \big\}\ D^{k+\nu}\big(Y(x)\big)\\ &= \{n\}_{(\mathbf{k})}\ (\alpha+\beta+1+n)_k\ Y(x)\,, \quad n\in\mathbb{N}.\end{aligned}}$$

The determination of the two sets of coefficients $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ and $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ realising the conditions (2.2.41)-(2.2.42) for this case is analogous to the corresponding problem in the Bessel case. In turn, the relation (2.2.64), with $A = \alpha+\beta+1$, yields

$$\Xi_{n-k}(k) = C^k_{\alpha,\beta}\ \Big(n(n+\alpha+\beta+1)\Big)_{\underline{k;\alpha+\beta+1}}\,, \quad n\in\mathbb{N},$$

and (2.2.51) permits to write

$$\begin{aligned}\Xi_{n-k}(k) &= C^k_{\alpha,\beta}\sum_{\nu=0}^{k}\ \widehat{s}_{\alpha+\beta+1}(k,\nu)\ \big(n(n+\alpha+\beta+1)\big)^{\nu}\\ &= C^k_{\alpha,\beta}\sum_{\nu=0}^{k}\ \widehat{s}_{\alpha+\beta+1}(k,\nu)\ \big(\chi_n\big)^{\nu}\,, \quad n\in\mathbb{N}\,.\end{aligned}$$

whence we obtain (2.2.41) with

$$d_{k,\tau} = C^k_{\alpha,\beta}\ \widehat{s}_{\alpha+\beta+1}(k,\tau)\,, \quad 0\leqslant\tau\leqslant k\,.$$

Conversely, due to (2.2.52) we have

$$\begin{aligned}(\chi_n)^k &= \big(n\,(n+\alpha+\beta+1)\big)^k = \sum_{\tau=0}^{k}\ \widehat{S}_{\alpha+\beta+1}(k,\tau)\ \Big(n\,(n+\alpha+\beta+1)\Big)_{\underline{\tau;\alpha+\beta+1}}\\ &= \sum_{\tau=0}^{k}\big(C(\tau;\alpha,\beta)\big)^{-1}\ \widehat{S}_{\alpha+\beta+1}(k,\tau)\ \Xi_{n-\tau}(\tau)\,, \quad n\in\mathbb{N}.\end{aligned}$$

The first and last members of the previous equality correspond to (2.2.42) if we consider

$$\widetilde{d}_{k,\tau} = \big(C(\tau;\alpha,\beta)\big)^{-1}\ \widehat{S}_{\alpha+\beta+1}(k,\tau)\,, \quad 0\leqslant\tau\leqslant k.$$

From corollary 2.2.5 it follows

$$\left\{\begin{aligned}&\sum_{\nu=0}^{k}\Lambda_{\nu}\,(k;x)\,D^{k+\nu} = \sum_{\tau=0}^{k}C^k_{\alpha,\beta}\ \widehat{s}_{\alpha+\beta+1}(k,\tau)\ \mathcal{F}^{\tau}\\ &\mathcal{F}^k = \sum_{\tau=0}^{k}\big(C(\tau;\alpha,\beta)\big)^{-1}\ \widehat{S}_{\alpha+\beta+1}(k,\tau)\ \sum_{\nu=0}^{\tau}\Lambda_{\nu}\,(\tau;x)\,D^{\tau+\nu}\end{aligned}\right.\,, \qquad (2.2.72)$$

where $\Lambda_{\nu}(k;x)$ is given by (2.2.71) and $\mathcal{F} = (x^2-1)D^2 + \big\{(\alpha+\beta+2)x - (\alpha-\beta)\big\}D$.

**Remark 2.2.4.**
It might be worthy to turn the attention to a well known result that is, in particular, presented by Comtet [30], Riordan [95, 96] but mostly developed in [96, chapter VI]. Consider the differential operator $\theta = x\,D$. It is possible to relate the powers[2] of $\theta$ and its "factorials", say $\theta_j = x^j D^j$, $j \in \mathbb{N}$, through the following equalities:

$$\theta^k = \big(x\,D\big)^k = \sum_{j=0}^{k} S(k,j)\, x^j\, D^j = \sum_{j=0}^{k} S(k,j)\,\theta_j\ , \quad k \in \mathbb{N},$$

$$\theta_k = x^k\, D^k = \sum_{j=0}^{k} s(k,j)\, \big(x\,D\big)^j = \sum_{j=0}^{k} s(k,j)\ \theta^j\ , \quad k \in \mathbb{N}.$$

The achieved relations (2.2.66), (2.2.68) ,(2.2.70) and (2.2.72) resemble the "inverse" formula just mentioned about the powers of $\theta$ and its factorials. Hence, representing by $\mathcal{F}_\tau := \sum_{\nu=0}^{\tau} \Lambda_\nu(\tau;x) D^{\tau+\nu}$, we have explicitly determined for each classical family two sets of coefficients $\{d_{k,\tau}\}_{0\leqslant\tau\leqslant k}$ and $\{\widetilde{d}_{k,\tau}\}_{0\leqslant\tau\leqslant k}$ such that

$$\mathcal{F}_k = \sum_{\tau=0}^{k} d_{k,\tau}\ \mathcal{F}^\tau \qquad \text{and} \qquad \mathcal{F}^k = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau}\ \mathcal{F}_\tau\ .$$

It appears indeed to be natural to view $\mathcal{F}_\tau$ as the $\tau^{\text{th}}$-factorial of Bochner's operator $\mathcal{F}$ and yet (2.2.66), (2.2.68) ,(2.2.70) and (2.2.72) are nothing but inverse relations between powers of Bochner's operator and its factorials.

**Remark 2.2.5.** The so-called $A$-modified Stirling numbers introduced in section 2.2.3, could also be called *Bessel-Stirling* numbers or *Jacobi-Stirling* numbers depending on the context and the values of the complex parameter $A$. Actually, in a recent work, Everitt et al. [45] have dealt with powers of Bochner's operator in the case of Jacobi classical family and within this context they have already used the name *Jacobi-Stirling* numbers when referring to the $(\alpha+\beta+1)$-modified Stirling numbers of first and second kind, here denoted as $\widehat{s}_{\alpha+\beta+1}(k,\nu)$ and $\widehat{S}_{\alpha+\beta+1}(k,\nu)$, respectively. In previous works, Everitt et al. [43, 44] have called to $\widehat{s}_1(k,\nu)$ and $\widehat{S}_1(k,\nu)$ as the Legendre-Stirling numbers of first and second kind[3], since Legendre polynomials correspond to a specialisation of Jacobi polynomials with $\alpha = \beta = 0$. However these same numbers could actually be viewed as the $(1)$-Bessel-Stirling numbers, inasmuch as they permit to establish "inverse relations" between any power of the Bochner operator associated to the

[2] To be more precise, the $k$-th power of $\theta$ is defined according to $\theta^k = (xD)^k = xD(xD)^{k-1}$, $k \in \mathbb{N}^*$, with the convention $(xD)^0 := \mathbb{I}$

[3] The sequence of Legendre-Stirling has already an entry at the OEIS, *cf.* entry A071951 in [100].

Bessel polynomials of parameter $\alpha = 1$ and the corresponding "factorials". In Table 2.6 (p.69) are listed the first $1$-*modified Stirling numbers*. Another good example lies on the $0$-modified Stirling numbers, that is $\widehat{s}_0(k,\nu)$ and $\widehat{S}_0(k,\nu)$ which indeed are connected to the Tchebyshev polynomials of first kind or also to the Bessel polynomials with parameter $\alpha = 1/2$. Anyway, the $0$-modified Stirling numbers (which could apparently be called the (first kind)Tchebyshev-Stirling numbers or the $(1/2)$-Bessel-Stirling numbers) are already known as the "*central factorial numbers*", just as it might be read in Riordan's book [96, pp. 212-217] (where we find $\widehat{s}_0(k,\nu) = t(2k, 2k-2\nu)$ and $\widehat{S}_0(k,\nu) = T(2k, 2k-2\nu)$) or in the entry A036969 of OEIS [100]. In Table 2.5 (p.68) are listed the first $0$-*modified Stirling numbers*.

To sum up, all these examples, Jacobi-Stirling, Legendre-Stirling are mere examples of the so-called $A$-modified Stirling numbers. Regarding this point of view, such specialisation of the $A$-modified Stirling numbers should be avoided, for the same reason that we do not use *Hermite*-Stirling or *Laguerre*-Stirling.

The information presented in Tables 2.3, 2.4, 2.5 and 2.6 is a result of computations made in *Mathematica*©, in accordance with:

```
StirlMod2[A_][0][0] := 1
StirlMod2[A_][0][j_] := KroneckerDelta[0, j]
StirlMod2[A_][n_][0] := KroneckerDelta[n, 0]
StirlMod2[A_][n_][j_] := StirlMod2[A][n][j]
    = StirlMod2[A][n - 1][j - 1] + j (j + A)*StirlMod2[A][n - 1][j]
```

and

```
stirlMod1[A_][0][0] := 1
stirlMod1[A_][0][j_] := KroneckerDelta[0, j]
stirlMod1[A_][n_][0] := KroneckerDelta[n, 0]
stirlMod1[A_][n_][j_] := stirlMod1[A][n][j]
  = stirlMod1[A][n - 1][j - 1] - (n - 1) (n - 1 + A)* stirlMod1[A][n - 1][j]
```

Table 2.3: A list of the first **A-modified Stirling numbers of $1^{\text{st}}$ kind**: $\widehat{s}_A(k,\nu)$, with $1 \leqslant \nu, k \leqslant 5$.

| $k \setminus \nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | $-(1+A)$ | 1 | 0 | 0 | 0 | 0 | 0 |
| 3 | $2\,(1+A)_2$ | $-5-3A$ | 1 | 0 | 0 | 0 | 0 |
| 4 | $-6\,(1+A)_3$ | $49+A(48+11A)$ | $-2(7+3A)$ | 1 | 0 | 0 | 0 |
| 5 | $24\,(1+A)_4$ | $-2(410+515\,A)$ $-2\,A^2(202+25A)$ | $273+5A(40+7A)$ | $-10(3+A)$ | 1 | 0 | 0 |
| 6 | $-120\,(1+A)_5$ | $2A(A(A(137A+1755)$ $+8045)+15525)+21076$ | $-A(A(225A+2279)$ $+7395)-7645$ | $1023+600A$ $+85A^2$ | $-5(3A+11)$ | 1 | 0 |
| 7 | $720\,(1+A)_6$ | $-36(A(A(A(A(49A+909)$ $+6475)+22015)$ $+35476)+21476)$ | $4(A(A(7A(58A+903)$ $+35626)+85785)$ $+74074)$ | $-7A(A(105A+1277)$ $+5019)-44473$ | $7(5A(5A+42)$ $+429)$ | -7 (3 A+13) | 1 |

Table 2.4: A list of the first **A-modified Stirling numbers of** $2^{\text{nd}}$ **kind**: $\widehat{S}_A(k,\nu)$, with $1 \leqslant \nu, k \leqslant 7$.

| $k \setminus \nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | $1+A$ | 1 | 0 | 0 | 0 | 0 | 0 |
| 3 | $(1+A)^2$ | $5+3A$ | 1 | 0 | 0 | 0 | 0 |
| 4 | $(1+A)^3$ | $21+A(24+7A)$ | $14+6A$ | 1 | 0 | 0 | 0 |
| 5 | $(1+A)^4$ | $(5+3A)(17+A(18+5A))$ | $147+5A(24+5A)$ | $10(3+A)$ | 1 | 0 | 0 |
| 6 | $(1+A)^5$ | $A(A(A(31A+222)$ $+604)+738)+341$ | $2(A(A(45A+332)$ $+831)+704)$ | $5A(13A+80)+627$ | $15A+55$ | 1 | 0 |
| 7 | $(1+A)^6$ | $(3A+5)(3A(A+4)+13)$ $\cdot(A(7A+24)+21)$ | $A(A(7A(43A+432)$ $+11566)+19920)+13013$ | $2(7A(A(25A+236)$ $+755)+5720)$ | $14(5A(2A+15)$ $+143)$ | $21A+91$ | 1 |

Table 2.5: A list of the first **0-modified Stirling numbers of** $1^{\text{st}}$ **kind** (also called *Central Factorials*)

| $k \diagdown \nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | -4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 36 | -13 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | -576 | 244 | -29 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 14400 | -6676 | 969 | -54 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | -518400 | 254736 | -41560 | 2913 | -90 | 1 | 0 | 0 | 0 | 0 |
| 7 | 25401600 | -13000464 | 2291176 | -184297 | 7323 | -139 | 1 | 0 | 0 | 0 |
| 8 | -1625702400 | 857431296 | -159635728 | 14086184 | -652969 | 16219 | -203 | 1 | 0 | 0 |
| 9 | 131681894400 | -71077637376 | 13787925264 | -1300616632 | 66976673 | -1966708 | 32662 | -284 | 1 | 0 |

and **0-modified Stirling numbers of** $2^{\text{nd}}$ **kind** (also called *Central Factorials*)

| $k \diagdown \nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 5 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 21 | 14 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 85 | 147 | 30 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | 1 | 341 | 1408 | 627 | 55 | 1 | 0 | 0 | 0 | 0 |
| 7 | 1 | 1365 | 13013 | 11440 | 2002 | 91 | 1 | 0 | 0 | 0 |
| 8 | 1 | 5461 | 118482 | 196053 | 61490 | 5278 | 140 | 1 | 0 | 0 |
| 9 | 1 | 21845 | 1071799 | 3255330 | 1733303 | 251498 | 12138 | 204 | 1 | 0 |
| 10 | 1 | 87381 | 9668036 | 53157079 | 46587905 | 10787231 | 846260 | 25194 | 285 | 1 |

Table 2.6: A list of the first **1-modified Stirling numbers of $1^{st}$ kind** (also called *Legendre-Stirling* of $1^{st}$ kind)

| $k \diagdown \nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | -6 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 72 | -18 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | -1440 | 432 | -38 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 43200 | -14400 | 1572 | -68 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | -1814400 | 648000 | -80424 | 4428 | -110 | 1 | 0 | 0 | 0 | 0 |
| 7 | 101606400 | -38102400 | 5151744 | -328392 | 10588 | -166 | 1 | 0 | 0 | 0 |
| 8 | -7315660800 | 2844979200 | -409027968 | 28795968 | -1090728 | 22540 | -238 | 1 | 0 | 0 |
| 9 | 658409472000 | -263363788800 | 39657496320 | -3000665088 | 126961488 | -3119328 | 43960 | -328 | 1 | 0 |

and **1-modified Stirling numbers of $2^{nd}$ kind** (also called *Legendre-Stirling* of $2^{nd}$ kind)

| $k \diagdown \nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 4 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 8 | 52 | 20 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 16 | 320 | 292 | 40 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | 32 | 1936 | 3824 | 1092 | 70 | 1 | 0 | 0 | 0 | 0 |
| 7 | 64 | 11648 | 47824 | 25664 | 3192 | 112 | 1 | 0 | 0 | 0 |
| 8 | 128 | 69952 | 585536 | 561104 | 121424 | 7896 | 168 | 1 | 0 | 0 |
| 9 | 256 | 419840 | 7096384 | 11807616 | 4203824 | 453056 | 17304 | 240 | 1 | 0 |
| 10 | 512 | 2519296 | 85576448 | 243248704 | 137922336 | 23232176 | 1422080 | 34584 | 330 | 1 |

## 2.3 Classical polynomials as a particular case of semiclassical polynomial sequences

The classical polynomials may be viewed as a special case of the so-called "**semiclassical polynomials**" introduced in the seminal paper of Shohat [99] and extensively studied by Pascal Maroni [79, 82, 84, 85, 88, 90].

**Definition 2.3.1.** A regular form $u \in \mathcal{P}'$ is said to be **semiclassical** if there is a monic polynomial $\Phi$ with $\deg \Phi = t \geqslant 0$ and a polynomial $\Psi$ with $\deg \Psi = p \geqslant 1$ such that $u$ fulfils

$$D(\Phi\, u) + \Psi\, u = 0\ . \tag{2.3.1}$$

Moreover, when $p = t-1$, necessarily $D^p\Psi(x) \neq n\ p!,\ n \in \mathbb{N}$. In this case, the associated MOPS to $u$ is said to be a **semiclassical polynomial sequence**.

Within the context of the previous definition, the pair of polynomials $(\Phi, \Psi)$ is not unique, regarding the fact that $u$ is also solution of $D(\chi\,\Phi\ u) + (\chi\,\Psi - \chi'\,\Phi)\ u = 0$ for any $\chi \in \mathcal{P}$. The equation (2.3.1) may be simplified if and only if there is a root $\xi$ of $\Phi$ such that

$$\begin{cases} \Phi'(\xi) + \Psi(\xi) = 0 \\ \langle u, \vartheta_\xi^2(\Phi) + \vartheta_\xi(\Psi)\rangle = 0 \end{cases}$$

and in this case, $u$ fulfils the equation

$$\Big(\vartheta_\xi(\Phi)\ u\Big)' + \Big\{\vartheta_\xi^2(\Phi) + \vartheta_\xi(\Psi)\Big\}\ u = 0$$

The **class of the semiclassical form** $u$ corresponds to the integer given by

$$s := \min\big\{\max\big(\deg(\Phi) - 2, \deg(\Psi) - 1\big)\big\}$$

where the minimum is taken over all the possible pairs $(\Phi, \Psi) \neq (0, 0)$ satisfying (2.3.1). The pair $(\widehat{\Phi}, \widehat{\Psi})$ furnishing the class $s \geqslant 0$ of $u$ is unique (*cf.* [82]). In the case where $s = 0$, the semiclassical form $u$ is indeed a *classical* form (Hermite, Laguerre, Bessel or Jacobi) and necessarily $\deg(\widehat{\Phi}) \leqslant 2$, $\deg(\widehat{\Psi}) = 1$.

Any affine transformation leaves invariant the semiclassical character of a form, inasmuch as the shifted form $\widetilde{u} = (h_{a^{-1}} \circ \tau_{-b})\, u$, with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, of the semiclassical form $u$ fulfilling (2.3.1) fulfils the equation

$$D\Big(\widetilde{\Phi}\,\widetilde{u}\Big) + \widetilde{\Psi}\,\widetilde{u} = 0$$

where $\widetilde{\Phi}(x) = a^{-\deg(\Phi)}\Phi(ax+b)$ and $\widetilde{\Psi}(x) = a^{1-\deg(\Phi)}\Psi(ax+b)$ (Maroni [82, 85]).

As a matter of fact, we have $u = \big(\tau_b \circ h_a\big)\,\widetilde{u}$, therefore, based on the properties (1.2.2)-(1.2.3), for any polynomial $g(\cdot)$ we successively have

$$\begin{aligned} g(x)\,u &= g(x)\big(\tau_b \circ h_a\big)\,\widetilde{u} = \tau_b\Big(\big(\tau_{-b}\,g\big)\big(h_a\widetilde{u}\big)\Big) = \tau_b \circ h_a\Big[\Big((h_{a^{-1}} \circ \tau_{-b})g\Big)\widetilde{u}\Big] \\ &= \big(\tau_b \circ h_a\big)\Big[g(ax+b)\widetilde{u}\Big] \end{aligned}$$

and, recalling (1.2.4)-(1.2.5) we deduce

$$D(g(x)\,u) \quad = \quad \tau_b\,D\Big[h_a\left(g(ax+b)\widetilde{u}\right)\Big] = \tfrac{1}{a}\,\big(\tau_b \circ h_a\big)D\Big[g(ax+b)\widetilde{u}\Big]\,.$$

This latter enables to obtain from (2.3.1) the following

$$\big(\tau_b \circ h_a\big)\Big\{a^{-1}D\Big[\Phi(ax+b)\widetilde{u}\Big] + \Psi(ax+b)\widetilde{u}\Big\} = 0$$

and after the multiplication by $a^{1-\deg\Phi}$ we get

$$\big(\tau_b \circ h_a\big)\Big\{a^{-\deg(\Phi)}D\Big[\Phi(ax+b)\widetilde{u}\Big] + a^{1-\deg(\Phi)}\Psi(ax+b)\widetilde{u}\Big\} = 0$$

whence the result.

In the case where two equations having a semiclassical form $u$ as solution are known, it is possible to derive a third one which is indeed a simplification of the original ones. More precisely, we have the result:

**Lemma 2.3.2.** *[82, p.144] Consider a semiclassical form $u$ such that*

$$D\left(\Phi_1 u\right) + \Psi_1 u \;=\; 0\,, \tag{2.3.2}$$

$$D\left(\Phi_2 u\right) + \Psi_2 u \;=\; 0\,, \tag{2.3.3}$$

*where $\deg\Phi_i = t_i$ and $\deg\Psi_i = p_i$, for $i = 1,2$. If $\Phi$ is the highest common factor between $\Phi_1$ e $\Phi_2$, there exists a polynomial $\Psi$ such that*

$$D\left(\Phi u\right) + \Psi u = 0 \quad .$$

Considering in the previous result stronger assumptions over the expressions of the polynomials $\Phi_1$ and $\Phi_2$, it is possible to deduce whether or not $u$ is a classical form. Within this matter, we recall a result given by Maroni and da Rocha [86], which will be useful for the sequel. We now present a more accurate proof, which already exists but in an unpublished version of the same work, kindly supplied by the authors.

**Lemma 2.3.3.** *[86] Let $\{P_n\}_{n\geqslant 0}$ be a semi-classical sequence, orthogonal with respect to $u_0$. Suppose that $u_0$ fulfills the next two functional equations*

$$\begin{aligned} D\left(\Phi_1 u_0\right)+\Psi_1 u_0 &= 0 \\ D\left(\Phi_2 u_0\right)+\Psi_2 u_0 &= 0 \end{aligned} \tag{2.3.4}$$

*and there exists an integer $m \geqslant 0$ and four polynomials $E$, $F$, $G$, $H$ such that*

$$\begin{aligned} \Phi_1(x) &= E(x)P_{m+1}(x)+F(x)P_m(x), \\ \Phi_2(x) &= G(x)P_{m+1}(x)+H(x)P_m(x). \end{aligned} \tag{2.3.5}$$

*Let $\Delta$ be the determinant of the system (2.3.5)*

$$\Delta\left(x\right)=\left|\begin{array}{cc} E\left(x\right) & F\left(x\right) \\ G\left(x\right) & H\left(x\right) \end{array}\right|. \tag{2.3.6}$$

*Then if one of the following conditions is fulfilled, the form $u_0$ is classical:*

(a) $\exists i=1,2$, *such that* $\deg\left(\Psi_i\right)\leqslant\deg\left(\Phi_i\right)-1$ *and* $\deg\left(\Delta\right)=2$;

(b) $\exists i=1,2$, *such that* $\deg\left(\Psi_i\right)=\deg\left(\Phi_i\right)$ *and* $\deg\left(\Delta\right)=1$;

(c) $\exists i=1,2$, *such that* $\deg\left(\Psi_i\right)=\deg\left(\Phi_i\right)+1$ *and* $\deg\left(\Delta\right)=0$.

*Proof.* Applying the Cramer's rule to the system (2.3.5), we get that

$$\begin{aligned} \Delta_k(x)P_{m+1}(x) &= \begin{vmatrix} \Phi_1(x) & F(x) \\ \Phi_2(x) & H(x) \end{vmatrix} = \Phi_1(x)H(x)-\Phi_2(x)F(x), \\ \Delta_k(x)P_m(x) &= \begin{vmatrix} \Phi_1(x) & E(x) \\ \Phi_2(x) & G(x) \end{vmatrix} = \Phi_1(x)E(x)-\Phi_2(x)G(x), \quad m\geqslant 0\,. \end{aligned}$$

Since $\{P_n\}_{n\geqslant 0}$ is an OPS, $P_m$ and $P_{m+1}$ have no common zeros. As a result, any common factor of $\Phi_1$ and $\Phi_2$, is also a factor of $\Delta$. In particular, the highest common factor of $\Phi_1$ and $\Phi_2$, say $\Phi$, is a factor of $\Delta$. Hence, we may express these polynomials as

$$\Phi_i=\Phi\widetilde{\Phi}_i \quad (\text{with } i=1,2) \quad \text{and} \quad \Delta=\Phi\widetilde{\Delta}. \tag{2.3.7}$$

lemma 2.3.2 assures the existence of a polynomial, $\Psi$, such that $D(\Phi u_0)+\Psi u_0=0$. Moreover, in its proof we see that such a polynomial satisfies the equalities given by:

$$\widetilde{\Phi}_i\Psi=\Psi_i+\widetilde{\Phi}_i'\Phi\,, \quad i=1,2.$$

Analyzing the degrees of the polynomials presented in both sides of the previous equation, we get that

$$\deg(\Phi_i)+\deg(\Psi)-\deg(\Phi)=\max\{\deg(\Psi_i),\deg(\Phi_i)-1\}\,. \tag{2.3.8}$$

Since, by hypothesis, $u_0$ is a semiclassical form, then $\deg \Psi \geqslant 1$. Furthermore, if $\deg \Delta \leqslant 2$, necessarily $\deg \Phi \leqslant 2$. It suffices now to show that $\deg \Psi = 1$, which allows us to say that $u_0$ is a semiclassical form of class $s = 0$ (i.e. a classical form).

In the case a), we get that (2.3.8) becomes $\deg \Phi = \deg \Psi + 1$. It follows $\deg \Phi \geqslant 2$, then $\deg \Phi = 2$ and consequently $\deg \Psi = 1$. The form $u_0$ is either a Bessel or a Jacobi form.

In the case b), we have, from (2.3.8), $\deg \Psi = \deg \Phi$, hence $\deg \Phi \geqslant 1$. But, $\deg \Phi \leqslant 1$, therefore $\deg \Phi = 1$ and $\deg \Psi = 1$. It is the Laguerre case.

Finally, in case c), on account of (2.3.8), we get $\deg \Psi = \deg \Phi + 1$ with $\deg \Phi = 0$. It is the Hermite case. □

## 2.4 New results about the characterisation of the classical polynomials

Up until now, we have devoted our study to necessary differential conditions fulfilled by the elements of a classical sequence. From this on, we are mostly interested in finding the reciprocal conditions, permitting to get a characterisation of classical sequences or the associated classical forms.

### 2.4.1 Characterisation through any even order differential equation

When a MOPS is solution of a certain differential equation of even order, we cannot, in general, infer about the classical or semiclassical character of the sequence if some supplementary conditions are not taken into account. Krall [65] and Kwon et al. [69, 70] has already treated this problem. Here we believe to give a more iluminating proof.

**Theorem 2.4.1.** *[73] Let $k \geqslant 1$ be an integer and $\{P_n\}_{n\geqslant 0}$ be a MOPS whose any polynomial $P_{n+k}$, $n \geqslant 0$, fulfills the differential equation*

$$\sum_{\nu=0}^{k} \Lambda_\nu\left(k;x\right)\left(P_{n+k}\right)^{(k+\nu)}\left(x\right) = \Xi_n\left(k\right) P_{n+k}\left(x\right),\ n \geqslant 0, \tag{2.4.1}$$

*where*

$$\Lambda_\nu(k;x) = \sum_{\tau=k-\nu}^{k+\nu} \xi_\tau^\nu P_\tau(x), \tag{2.4.2}$$

*with* $\xi_{\tau}^{\nu} \in \mathbb{C}$ *and* $\xi_{k-\nu}^{\nu} \neq 0,\ 0 \leqslant \nu \leqslant k$ *and* $\Xi_n(k) \in \mathbb{C}\backslash\{0\}$.

*Then* $\{P_n\}_{n\geqslant 0}$ *is a classical sequence.*

*Proof.* Let $m$ be an integer such that $0 \leqslant m \leqslant k-1$. If we multiply both sides of (2.4.1) and, afterwards, we consider the action of $u_0$ over the resulting equation, then we get:

$$\left\langle u_0, \sum_{\nu=0}^{k} \Lambda_\nu(k;x) P_m(x) (P_{n+k})^{(k+\nu)}(x) \right\rangle = \left\langle u_0, \Xi_n(k) P_m(x) P_{n+k}(x) \right\rangle,\ n \geqslant 0. \tag{2.4.3}$$

Since $\{P_n\}_{n\geqslant 0}$ is a MOPS, from (2.4.3) we have

$$\left\langle \sum_{\nu=0}^{k} (-1)^{k+\nu} D^{k+\nu} \left(\Lambda_\nu(k;x) P_m(x) u_0\right), P_{n+k}(x) \right\rangle = 0, \quad n \geqslant 0. \tag{2.4.4}$$

It can be easily seen that:

$$\left\langle \sum_{\nu=0}^{k} (-1)^{k+\nu} D^{k+\nu} \left(\Lambda_\nu(k;x) P_m(x) u_0\right), P_j(x) \right\rangle = 0, \quad 0 \leqslant j \leqslant k-1, \tag{2.4.5}$$

due to the fact that $D^{k+\nu} P_j(x) = 0,\ 0 \leqslant j \leqslant k-1$.

Therefore, once $\{P_n\}_{n\geqslant 0}$ is a PS, (2.4.4) together with (2.4.5) imply that $u_0$ satisfies the following functional equations:

$$\sum_{\nu=0}^{k} (-1)^{\nu} D^{\nu} \left(\Lambda_\nu(k;x) P_m(x) u_0\right) = 0, \quad 0 \leqslant m \leqslant k-1. \tag{2.4.6}$$

For the sake of simplicity, let us write

$$\begin{aligned} \Lambda_\nu &= \Lambda_\nu(k;x), 0 \leqslant \nu \leqslant k, \\ P_n &= P_n(x), n \geqslant 0. \end{aligned}$$

By virtue of *Leibniz derivation formula* given on page 25, the system of $k$ functional equations given by (2.4.6) is equivalent to

$$\sum_{\mu=0}^{k} (P_m)^{(\mu)} \sum_{\nu=\mu}^{k} (-1)^{\nu} \binom{\nu}{\mu} D^{\nu-\mu} \left(\Lambda_\nu u_0\right) = 0, \quad 0 \leqslant m \leqslant k-1, \tag{2.4.7}$$

The goal is to simplify the system of equations (2.4.7) into one of $k$ differential equations of order one. This simplification can be done by means of lemmas 2.4.2 and 2.4.3, see below.

Thus, following lemma 2.4.2, (2.4.7) may be written as

$$\sum_{\nu=m}^{k} (-1)^{\nu} \binom{\nu}{m} D^{\nu-m} \left(\Lambda_{\nu} u_0\right) = 0, \quad 0 \leqslant m \leqslant k-1. \tag{2.4.8}$$

Now, in accordance with lemma 2.4.3, see below, (2.4.8) imply

$$(k-\mu) D(\Lambda_{k-\mu} u_0) - (\mu+1)\Lambda_{k-\mu-1} u_0 = 0, \quad 0 \leqslant \mu \leqslant k-1. \tag{2.4.9}$$

This means that $u_0$ is a semiclassical form. In particular, when we take $\mu = k-1$ and $\mu = k-2$ in (2.4.9), we have that $u_0$ satisfies the next two functional equations:

$$\begin{cases} D\left(\Lambda_1 u_0\right) + \left(-k\,\Lambda_0\right) u_0 = 0, \\ \\ D\left(\Lambda_2 u_0\right) + \left(-\frac{k-1}{2}\,\Lambda_1\right) u_0 = 0. \end{cases} \tag{2.4.10}$$

where the polynomials $\Lambda_{\nu},\ 0 \leqslant \nu \leqslant 2$, are given by

$$\begin{aligned} \Lambda_0 &= \xi_k^0 P_k, \\ \Lambda_1 &= \xi_{k+1}^1 P_{k+1} + \xi_k^1 P_k + \xi_{k-1}^1 P_{k-1}, \\ \Lambda_2 &= \xi_{k+2}^2 P_{k+2} + \xi_{k+1}^2 P_{k+1} + \xi_k^2 P_k + \xi_{k-1}^2 P_{k-1} + \xi_{k-2}^2 P_{k-2} \end{aligned} \tag{2.4.11}$$

Let us now consider $N_1\,\Phi_1 = \Lambda_1$ and $N_2\,\Phi_2 = \Lambda_2$, where $N_1$ and $N_2$ are two normalization constants. Thus, we may write (2.4.10) like

$$\begin{cases} D\left(\Phi_1 u_0\right) + \Psi_1 u_0 = 0, \\ \\ D\left(\Phi_2 u_0\right) + \Psi_2 u_0 = 0. \end{cases} \tag{2.4.12}$$

with

$$\Psi_1 = -k\,\left(N_1^{-1}\,\Lambda_0\right) = -k\,N_1^{-1}\,\xi_k^0\,P_k \tag{2.4.13}$$

and $\Psi_2 = -\frac{k-1}{2}\,\left(N_2^{-1}\,\Lambda_1\right)$. Since $\{P_n\}_{n\geqslant 0}$ is MOPS by virtue of (2.4.11), it is possible to write $\Psi_2$, $\Phi_1$ and $\Phi_2$ as

$$\begin{aligned} \Psi_2 &= -\frac{(k-1)N_2^{-1}}{2}\,\big(E_k P_{k+1} + F_k P_k\big), \\ \Phi_1 &= N_1^{-1}\big(E_k P_{k+1} + F_k P_k\big), \\ \Phi_2 &= N_2^{-1}\big(G_k P_{k+1} + H_k P_k\big), \end{aligned} \tag{2.4.14}$$

where

$$\begin{aligned}
E_k &= \xi^1_{k+1} - \frac{\xi^1_{k-1}}{\gamma_k},\\
F_k &= \left(\frac{\xi^1_{k-1}}{\gamma_k}\right) x + \left(\xi^1_k - \frac{\xi^1_{k-1}}{\gamma_k}\,\beta_k\right),\\
G_k &= \left(\xi^2_{k+2} - \frac{\xi^2_{k-2}}{\gamma_k\gamma_{k-1}}\right) x + \left(-\xi^2_{k+2}\beta_{k+1} + \frac{\xi^2_{k-2}}{\gamma_k\gamma_{k-1}}\,\beta_{k-1} + \xi^2_{k+1} - \frac{\xi^2_{k-1}}{\gamma_k}\right),\\
H_k &= \left(\xi^2_{k-2}\frac{1}{\gamma_{k-1}\gamma_k}\right) x^2 + \left(\xi^2_{k-1}\frac{1}{\gamma_k} + \xi^2_{k-2}\frac{1}{\gamma_{k-1}\gamma_k}(-\beta_{k-1}-\beta_k)\right) x\\
&\quad + \left(-\xi^2_{k+2}\gamma_{k+1} + \xi^2_k - \xi^2_{k-1}\frac{1}{\gamma_k}\beta_k + \xi^2_{k-2}\frac{1}{\gamma_{k-1}\gamma_k}\beta_{k-1}\beta_k - \xi^2_{k-2}\frac{1}{\gamma_{k-1}}\right).
\end{aligned}$$

If we denote by $\Delta_k$ the determinant of the last two equations of (2.4.14), that is,

$$\Delta_k(x) = \begin{vmatrix} E_k & F_k \\ G_k & H_k \end{vmatrix},$$

then, by hypothesis, $\deg(\Delta_k) \leqslant 2$. After some straightforward calculations, we can write $\Delta_k$ as

$$\Delta_k = \delta^2_k x^2 + \delta^1_k x + \delta^0_k,$$

where

$$\delta^2_k = \frac{1}{\gamma_k}\left\{\frac{\xi^2_{k-2}\xi^1_{k+1}}{\gamma_{k-1}} - \xi^2_{k+2}\xi_{k-1}\right\},$$

$$\begin{aligned}
\delta^1_k \;&=\; -(\beta_k+\beta_{k+1})\delta^2_k - \xi^1_k\xi^2_{k+2} + \tfrac{1}{\gamma_k}\left\{\xi^1_{k+1}\xi^2_{k-1} - \xi^1_{k-1}\xi^2_{k+1}\right\}\\
&\quad + \tfrac{1}{\gamma_k\gamma_{k-1}}\left\{\xi^1_k\xi^2_{k-2} + (\beta_{k+1}-\beta_{k-1})\xi^1_{k+1}\xi^2_{k-2}\right\}
\end{aligned}$$

$$\begin{aligned}
\delta^0_k \;&=\; -\beta_k\delta^1_k - ({\beta_k}^2+\gamma_{k+1})\delta^2_k - \gamma_{k+1}\xi^1_{k+1}\xi^2_{k+2} + \xi^1_{k+1}\xi^2_k - \xi^1_k\xi^2_{k+1}\\
&\quad + (\beta_{k+1}-\beta_k)\xi^1_k\xi^2_{k+2} + \tfrac{1}{\gamma_k}(\xi^1_k\xi^2_{k-1} - \xi^1_{k-1}\xi^2_k)\\
&\quad + \tfrac{1}{\gamma_k\gamma_{k-1}}(\xi^1_{k-1} + \gamma_{k+1}\xi^1_{k+1} + (\beta_k-\beta_{k-1})\xi^1_k)\xi^2_{k-2}
\end{aligned}$$

In accordance with (2.3.7) presented in the proof of lemma 2.3.3, we have that $\deg(\Phi) \leqslant \deg(\Delta_k)$. Thus, no matter which the expressions of the coefficients $\delta^i_k\,(i=0,1,2)$ are, we will always have $\deg \Phi \leqslant 2$. Yet, this is not sufficient to say that $u_0$ is a classical form. We will absolutely need to show that there exists a polynomial $\Psi$ such that $u_0$ fulfills $D(\Phi u_0) + \Psi u_0 = 0$ and $\deg \Psi = 1$. Actually, this can be done by making use of lemma 2.3.3. So, our analysis will consist on studying what happens when $\deg \Delta_k$ is equal to 2, 1 or 0.

Suppose that $\delta_k^2 \neq 0$, which implies that $\deg \Delta_k = 2$. If $\xi_{k+1}^1 \neq 0$, then $\deg \Phi_1 = k+1$ and $\deg \Psi_1 = k = \deg \Phi_1 - 1$, in accordance with (2.4.14) and (2.4.13). So, the condition (a) of lemma 2.3.3 is satisfied. On the other hand, if $\xi_{k+1}^1 = 0$, then, on account of (2.4.14), $\deg \Psi_2 \leqslant k$ and we will necessarily have $\xi_{k+2}^2 \neq 0$, due to $\delta_k^2 \neq 0$, which means that $\deg \Psi_2 \leqslant k \leqslant k+1 = \deg \Phi_2 - 1$. Once more, we are in the condition (a) of lemma 2.3.3. In both of these cases, $u_0$ is either a Bessel form or a Jacobi form.

Now, suppose that $\delta_k^2 = 0$ and $\delta_k^1 \neq 0$, that is, $\deg \Delta_k = 1$. We will necessarily have $\xi_{k+1}^1 = 0$. Otherwise, we would have, from (2.4.13), $\deg \Psi_1 = k$ and from (2.4.14) $\deg \Phi_1 = k+1$, so, on account of (2.3.8), this would imply $\deg \Psi = \deg \Phi - 1$, which contradicts the hypothesis $\deg \Phi \leqslant \deg \Delta_k \leqslant 1$, since the regularity conditions of $u_0$ imply $\deg \Psi \geqslant 1$, and therefore we would have $\deg \Phi \geqslant 2$. As a consequence, we will have $\xi_{k+1}^1 = \xi_{k+2}^2 = 0$. Under these conditions, the expression of $\delta_k^1$ becomes

$$\delta_k^1 = \frac{1}{\gamma_k}\left\{-\xi_{k-1}^1 \xi_{k+1}^2\right\} + \frac{1}{\gamma_k \gamma_{k-1}}\left\{\xi_k^1 \xi_{k-2}^2\right\} .$$

Actually, we will necessarily have $\xi_k^1 \neq 0$. If $\xi_k^1 = 0$, then $\xi_{k+1}^2 \neq 0$ (since $\delta_k^1 \neq 0$), and consequently, from (2.4.14), $\deg \Psi_2 = k-1$ and $\deg \Phi_2 = k+1$. As a result, the regularity conditions of $u_0$ ($\deg \Psi \geqslant 1$) together with (2.3.8), imply $\deg \Phi \geqslant 2$, which contradicts the hypothesis $\deg \Phi \leqslant \deg \Delta_k \leqslant 1$. Thus, $\deg \Psi_1 = k = \deg \Phi_1$, and lemma 2.3.3 assures that $u_0$ is a classical form. More precisely it is a Laguerre form.

To finalize our discussion, let us suppose that $\delta_k^2 = \delta_k^1 = 0$. Then $\Delta_k = \delta_k^0$ and the two following equalities hold:

$$\begin{aligned} &\xi_{k-2}^2 \xi_{k+1}^1 = \gamma_{k-1}\, \xi_{k+2}^2 \xi_{k-1}^1 \\ &\xi_{k-1}^1 \xi_{k+1}^2 = \xi_{k+1}^1 \xi_{k-1}^2 + \left(\tfrac{1}{\gamma_{k-1}} \xi_{k-2}^2 - \gamma_k \xi_{k+2}^2\right)\xi_k^1 - (\beta_{k-1} - \beta_{k+1})\xi_{k+1}^1 \xi_{k-2}^2 \end{aligned} \qquad (2.4.15)$$

On account of the previous discussion, necessarily, $\xi_{k+1}^1 = 0$, therefore, from (2.4.15), $\xi_{k+2}^2 = 0$. If we suppose $\xi_k^1 \neq 0$, then (2.4.13) and (2.4.14) would, respectively, imply $\deg \Psi_1 = k$ and $\deg \Phi_1 = k$. Therefore $\deg \Psi = \deg \Phi = 0$, due to (2.3.8). But this contradicts the regularity condition of $u_0$: $\deg \Psi \geqslant 1$. So $\xi_k^1 = 0$. One has $\deg \Phi_1 = k-1$, thus $\deg \Psi = \deg \Phi + 1 = 1$, it is the Hermite case. On the other hand $\deg \Psi_2 = k-1$ and $\deg \Phi_2 \leqslant k$, since $\xi_{k+1}^2 = 0$. But $\deg \Phi_2 = k$ implies $\deg \Psi = \deg \Phi - 1 = -1$ which is not possible. Consequently, $\xi_k^2 = 0$

and $\delta_k^0 \neq 0$, since $\delta_k^0 = \gamma_{k-1}^{-1}\gamma_k^{-1}\xi_{k-1}^1\xi_{k-2}^2$. Now, lemma 2.3.3 allows us to conclude that, in this case, $u_0$ is a Hermite classical form. □

To end this section, we present the two lemmas needed to the completion of the previous proof.

**Lemma 2.4.2.** *[73] The system of $k$ equations given by*

$$\sum_{\mu=0}^{k} (P_m)^{(\mu)} \sum_{\nu=\mu}^{k} (-1)^\nu \binom{\nu}{\mu} D^{\nu-\mu}(\Lambda_\nu u_0) = 0, \quad 0 \leqslant m \leqslant k-1, \tag{2.4.16}$$

*is equivalent to*

$$\sum_{\nu=m}^{k} (-1)^\nu \binom{\nu}{m} D^{\nu-m}(\Lambda_\nu u_0) = 0, \quad 0 \leqslant m \leqslant k-1. \tag{2.4.17}$$

*Proof.* We begin with the proof that (2.4.16) implies (2.4.17). For $m=0$, (2.4.16) becomes

$$\sum_{\nu=0}^{k} (-1)^\nu \binom{\nu}{0} D^{\nu}(\Lambda_\nu u_0) = 0.$$

For $1 \leqslant m \leqslant k-1$ $(k \geqslant 2)$, suppose that

$$\sum_{\nu=\mu}^{k} (-1)^\nu \binom{\nu}{\mu} D^{\nu-\mu}(\Lambda_\nu u_0) = 0, \quad 0 \leqslant \mu \leqslant m-1.$$

Since $(P_m)^{(\mu)}(x) = 0,\ \mu \geqslant m+1$ and $(P_m)^{(m)}(x) = m!$, we have

$$m! \sum_{\nu=m}^{k} (-1)^\nu \binom{\nu}{m} D^{\nu-m}(\Lambda_\nu u_0) + \sum_{\mu=0}^{m-1} (P_m)^{(\mu)} \sum_{\nu=\mu}^{k} (-1)^\nu \binom{\nu}{\mu} D^{\nu-\mu}(\Lambda_\nu u_0) = 0.$$

Therefore,

$$\sum_{\nu=m}^{k} (-1)^\nu \binom{\nu}{m} D^{\nu-m}(\Lambda_\nu u_0) = 0.$$

It is evident that (2.4.17) implies (2.4.16). □

The next lemma shows that it is possible to transform (2.4.17) into a system of $k$ differential functional equations of order one.

**Lemma 2.4.3.** *[73] If a form $u_0$ fulfills the $k$ equations given by (2.4.17), then it also fulfills the following $k$ equations:*

$$(k-\mu)D(\Lambda_{k-\mu}u_0)-(\mu+1)\Lambda_{k-\mu-1}u_0=0, \quad 0\leqslant\mu\leqslant k-1. \tag{2.4.18}$$

*Proof.* If we take $m=k-1$ in (2.4.17), we naturally have

$$kD(\Lambda_k u_0)-\Lambda_{k-1}u_0=0.$$

Thus, (2.4.18) is valid for $\mu=0$.

When $1\leqslant\mu\leqslant k-2$, we suppose that (2.4.18) holds for $0\leqslant\nu\leqslant\mu$:

$$(k-\mu+\nu)D(\Lambda_{k-\mu+\nu}u_0)=(\mu-\nu+1)\Lambda_{k-\mu+\nu-1}u_0 \quad . \tag{2.4.19}$$

Now, for $m=k-\mu-2$, it is possible to write (2.4.17) as

$$(-1)^{k-\mu-2}\Lambda_{k-\mu-2}u_0+(-1)^{k-\mu-1}(k-\mu-1)D(\Lambda_{k-\mu-1}u_0)+S_\mu=0 \quad , \tag{2.4.20}$$

where

$$S_\mu=\sum_{\nu=k-\mu}^{k}(-1)^\nu\binom{\nu}{k-\nu-2}D^{\nu-k+\mu+2}(\Lambda_\nu u_0),$$

*i.e.* ,

$$S_\mu=\sum_{\nu=0}^{\mu}(-1)^{k-\mu+\nu}\binom{k-\mu+\nu}{\nu+2}D^{\nu+2}(\Lambda_{k-\mu+\nu}u_0). \tag{2.4.21}$$

We shall be transforming $S_\mu$. Indeed, we get

$$\begin{aligned} S_\mu &= (-1)^{k-\mu}\binom{k-\mu}{2}D^2\left(\Lambda_{k-\mu}u_0\right)\\ &+\sum_{\nu=1}^{\mu}(-1)^{k-\mu+\nu}\binom{k-\mu+\nu}{\nu+2}D^2\left(\frac{\mu-\nu+1}{k-\mu+\nu}\,D^{\nu-1}\left(\Lambda_{k-\mu+\nu-1}u_0\right)\right) \quad , \end{aligned}$$

since from (2.4.19) we have

$$(k-\mu+\nu)D^\nu(\Lambda_{k-\mu+\nu}u_0)=(\mu-\nu+1)D^{\nu-1}(\Lambda_{k-\mu+\nu-1}u_0), \quad \nu\geqslant 1 \quad . \tag{2.4.22}$$

Therefore,

$$S_\mu=(-1)^{k-\mu}\binom{k-\mu}{2}D^2(\Lambda_{k-\mu}u_0)+\sum_{\nu=0}^{\mu-1}(-1)^{k-\mu+\nu}\binom{k-\mu+\nu}{\nu+2}D^{\nu+2}(\Lambda_{k-\mu+\nu}u_0)$$

$$\begin{aligned} &=(-1)^{k-\mu}\left(1-\frac{\mu}{3}\right)\binom{k-\mu}{2}D^2(\Lambda_{k-\mu}u_0)\\ &+\sum_{\nu=0}^{\mu-2}(-1)^{k-\mu+\nu+2}\binom{k-\mu+\nu+2}{\nu+4}\frac{(\mu-\nu-1)(\mu-\nu)}{(k-\mu+\nu+2)(k-\mu+\nu+1)}\,D^{2+\nu}(\Lambda_{k-\mu+\nu}u_0)\,. \end{aligned}$$

We are proceeding by induction. Suppose that

$$S_\mu = (-1)^{k-\mu}\, a_{\tau-1}(\mu)\binom{k-\mu}{2} D^2(\Lambda_{k-\mu}u_0) + \sum_{\nu=0}^{\mu-\tau}(-1)^{k-\mu+\nu+\tau}\binom{k-\mu+\nu+\tau}{\nu+\tau+2}\prod_{\xi=0}^{\tau-1}\frac{\mu-\nu-\xi}{k-\mu+\nu+\xi+1}\, D^{2+\nu}(\Lambda_{k-\mu+\nu}u_0), \qquad 1\leqslant\tau\leqslant\mu-1, \tag{2.4.23}$$

with $a_0(\mu)=1$. As above, we have

$$\begin{aligned} S_\mu = {} & (-1)^{k-\mu}\Big\{a_{\tau-1}(\mu)\binom{k-\mu}{2} + (-1)^\tau\binom{k-\mu+\tau}{\tau+2}\prod_{\xi=0}^{\tau-1}\frac{\mu-\xi}{k-\mu+\xi+1}\Big\} D^2\left(\Lambda_{k-\mu}u_0\right) \\ & + \sum_{\nu=1}^{\mu-\tau}\left\{(-1)^{k-\mu+\nu+\tau}\binom{k-\mu+\nu+\tau}{\nu+\tau+2}\prod_{\xi=0}^{\tau-1}\left(\frac{\mu-\nu-\xi}{k-\mu+\nu+\xi+1}\right)\right. \\ & \left. D^2\!\left(\frac{\mu-\nu+1}{k-\mu+\nu}\, D^{\nu-1}\left(\Lambda_{k-\mu+\nu-1}u_0\right)\right)\right\}, \end{aligned}$$

if we take (2.4.23) into account. Consequently,

$$\begin{aligned} S_\mu = {} & (-1)^{k-\mu}a_\tau(\mu)\binom{k-\mu}{2} D^2\left(\Lambda_{k-\mu}u_0\right) \\ & + \sum_{\nu=0}^{\mu-\tau-1}\left\{(-1)^{k-\mu+\nu+\tau+1}\binom{k-\mu+\nu+\tau+1}{\nu+\tau+3}\prod_{\xi=0}^{\tau}\left(\frac{\mu-\nu-\xi}{k-\mu+\nu+\xi+1}\right)\right. \\ & \left. D^{\nu+2}\left(\Lambda_{k-\mu+\nu}u_0\right)\right\}, \end{aligned}$$

where

$$\binom{k-\mu}{2}a_\tau(\mu) = \binom{k-\mu}{2}a_{\tau-1}(\mu) + (-1)^\tau\binom{k-\mu+\tau}{\tau+2}\prod_{\xi=0}^{\tau-1}\frac{\mu-\xi}{k-\mu+\xi+1} \quad .$$

But

$$\binom{k-\mu+\tau}{\tau+2}\prod_{\zeta=0}^{\tau-1}\frac{\mu-\zeta}{k-\mu+\zeta+1} = \binom{k-\mu}{2}\binom{\mu}{\tau}\frac{2}{(\tau+1)(\tau+2)}\,,$$

whence

$$a_\tau(\mu) - a_{\tau-1}(\mu) = (-1)^\tau\binom{\mu}{\tau}\frac{2}{(\tau+1)(\tau+2)} \quad . \tag{2.4.24}$$

It follows,

$$a_\tau(\mu) = 1 + \sum_{\nu-1}^{\tau}(-1)^\tau\binom{\mu}{\nu}\frac{2}{(\nu+1)(\nu+2)} \quad .$$

As a result, we deduce, in the particular case of $\tau = \mu$, that

$$a_\mu(\mu) = \sum_{\tau=0}^{\mu} (-1)^\tau \binom{\mu}{\tau} \frac{2}{(\tau+1)(\tau+2)} \ , \quad \mu \geqslant 0.$$

Besides, if we consider the following relation

$$(1-x)^\mu = \sum_{\tau=0}^{\mu} \binom{\mu}{\tau} (-1)^\tau x^\tau \ ,$$

after two integrations, we get:

$$\frac{1}{\mu+1} \left\{ x + \frac{(1-x)^{\mu+2} - 1}{\mu+2} \right\} = \sum_{\tau=0}^{\mu} \binom{\mu}{\tau} (-1)^\tau \frac{x^{\tau+2}}{(\tau+1)(\tau+2)}.$$

Taking $x = 1$ in the previous relation, we find

$$a_\mu(\mu) = \frac{2}{\mu+2} \quad .$$

Now, taking $\tau = \mu$ in (2.4.23), we obtain

$$S_\mu = (-1)^{k-\mu} \binom{k-\mu}{2} a_\mu(\mu) D^2(\Lambda_{k-\nu} u_0) \quad ,$$

on account of (2.4.24). Finally, (2.4.20) becomes

$$\Lambda_{k-\mu-2}\, u_0 - (k-\mu-1)\, D(\Lambda_{k-\mu-1} u_0) + \binom{k-\mu}{2} a_\mu(\mu)\, D^2(\Lambda_{k-\mu} u_0) = 0 \quad .$$

As long as

$$(k-\mu)\, D^2(\Lambda_{k-\mu}\, u_0) = (\mu+1) D(\Lambda_{k-\mu-1}\, u_0) \quad ,$$

we conclude that

$$(k-\mu-1)\, D(\Lambda_{k-\mu-1}\, u_0) - \left(1 - \frac{\mu+1}{2} a_\mu(\mu)\right)^{-1} \Lambda_{k-\mu-2}\, u_0 = 0$$

which is (2.4.17) where $\mu \to \mu + 1$ . □

### 2.4.2 Generalised Rodrigues-type formula

The classical polynomials may be characterised through the *Rodrigues-type formula* (2.0.4) and its analogous relation was also mentioned (see formula (2.0.6)). The next result characterises the classical polynomials by means of a generalisation of the *Rodrigues*-type formula.

**Proposition 2.4.4.** *A given MOPS $\{P_n\}_{n\in\mathbb{N}}$ with respect to the regular form $u_0$ is classical if and only if there is a monic polynomial $\Phi$, with $\deg\Phi \leqslant 2$, such that*

$$D^k\Big(\lambda_0^k\ \vartheta_k\ \Phi^k(x)\ \big(D^k P_{n+k}\big)(x)\ u_0\Big) = (n+1)_k\ \lambda_n^k\ P_{n+k}(x)\ u_0\ , n\in\mathbb{N}, \tag{2.4.25}$$

*holds for any positive integer $k$, where $\lambda_n^k$ is given by (2.2.4) and $\vartheta_k \neq 0$.*

*Proof.* Suppose $\{P_n\}_{n\in\mathbb{N}}$ is a classical MOPS and $u_0$ the associated classical form. Therefore, the sequence $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is also a MOPS for each integer $k \geqslant 1$ and there is a sequence of nonzero numbers $\{\vartheta_n\}_{n\in\mathbb{N}}$ and a monic polynomial $\Phi$ with $\deg\Phi \leqslant 2$ such that the identity (2.0.4) holds. Besides, from corollary 2.1.1, we further have that $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is a classical sequence and its associated classical form may be expressed as $u_0^{[k]} = \zeta_k\ \Phi^k(x)\, u_0$ with $\zeta_k = \lambda_0^k\ \vartheta_k$, where $\lambda_0^k$ is given by (2.2.4). Under the assumptions it is clear that (2.2.8) holds, which may be expressed like

$$D^k\Big(\lambda_0^k\ \vartheta_k\ \Phi^k(x)\ P_n^{[k]}(x)\ u_0\Big) = \lambda_n^k\ P_{n+k}\ u_0, \quad n\in\mathbb{N}.$$

Following the definition of $\{P_n^{[k]}\}_{n\in\mathbb{N}}$, the previous equality yields (2.4.25).

Conversely, suppose $\{P_n\}_{n\in\mathbb{N}}$ is a MOPS with respect to the regular form $u_0$ and there is a monic polynomial $\Phi$, with $\deg\Phi \leqslant 2$ such that (2.4.25) holds for any integer $k \geqslant 1$, with $\lambda_n^k \neq 0$, $n\in\mathbb{N}$, and $\vartheta_k \neq 0$. Based on the Leibniz derivation formula for derivation, it is possible to transform (2.4.25) into

$$\sum_{\tau=0}^{k}\binom{k}{\tau}\big(D^{k+\tau}P_{n+k}\big)(x)\ D^{k-\tau}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big) = \Xi_n(k)\ P_{n+k}(x)\ u_0, \tag{2.4.26}$$

for any $n\in\mathbb{N}$. In particular, this last equality also holds for any integer $m$ such that $0 \leqslant m \leqslant k$, and because we have $\big(D^{k+\tau}P_{k+m}\big)(x) = 0$ when $\tau \geqslant m+1$, we are able to write (2.4.26) like:

$$\sum_{\tau=0}^{m}\binom{k}{\tau}\big(D^{k+\tau}P_{n+k}\big)(x)\ D^{k-\tau}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big) = \Xi_n(k)\ P_{n+k}(x)\ u_0,\ 0\leqslant m\leqslant k. \tag{2.4.27}$$

The particular choice of $m=0$ in the previous equality brings

$$k!\ D^k\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big) = \Xi_0(k)\ P_k(x)\ u_0\ . \tag{2.4.28}$$

By taking $m=1$ in (2.4.27) and on account of (2.4.28), we obtain

$$D^{k-1}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big) = S_{k+1}(x)\ u_0, \tag{2.4.29}$$

where

$$S_{k+1}(x) = \frac{1}{k\,(k+1)!}\,\left\{\Xi_1(k)P_{k+1}(x) - \frac{1}{k!}\,\Xi_0(k)\,\big(D^kP_{k+1}\big)(x)\,P_k(x)\right\}\,.$$

Since $\deg\big(D^kP_{k+1}\big)=1$ and $\{P_n\}_{n\in\mathbb{N}}$ is a MOPS satisfying a second order recurrence relation of the type (1.4.3), there is a set of coefficients $\{\xi_\nu^{k+1}\}_{k-1\leqslant\nu\leqslant k+1}$, with $\xi_{k-1}^{k+1}\neq 0$, permitting to write $S_{k+1}(x)=\sum\limits_{\nu=k-1}^{k+1}\xi_\nu^{k+1}P_\nu(x)$. Now, assume that for $0\leqslant\mu\leqslant k-1$ there is a set of coefficients $\{\xi_\nu^{k+\mu}\}_{k-\mu\leqslant\nu\leqslant k+\mu}$ with $\xi_{k-\mu}^{k+\mu}\neq 0$, such that

$$D^{k-\mu}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big) = S_{k+\mu}(x)u_0 \tag{2.4.30}$$

where

$$S_{k+\mu}(x) = \sum_{\nu=k-\mu}^{k+\mu}\xi_\nu^{k+\mu}\,P_\nu(x)\,. \tag{2.4.31}$$

The relation (2.4.27), which may be equivalently expressed as follows

$$\begin{aligned}&\binom{k}{m}(k+m)!D^{k-m}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big)\\&+\sum_{\tau=0}^{m-1}\binom{k}{\tau}\big(D^{k+\tau}P_{k+m}\big)(x)\;D^{k-\tau}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big)=\Xi_m(k)P_{k+m}\,u_0,\end{aligned}$$

becomes, under the assumption like

$$\begin{aligned}&\binom{k}{m}(k+m)!D^{k-m}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big)\\&=\left\{\Xi_m(k)P_{k+m}-\sum_{\tau=0}^{m-1}\binom{k}{\tau}\big(D^{k+\tau}P_{k+m}\big)(x)\sum_{\nu=k-\tau}^{k+\tau}\xi_\nu^{k+\tau}\,P_\nu(x)\right\}u_0\,.\end{aligned} \tag{2.4.32}$$

Based on the second order recurrence relation fulfilled by the MOPS $\{P_n\}_{n\in\mathbb{N}}$ and also on the fact that $(D^{k+\tau}P_{k+m})$ is a $(m-\tau)$-degree polynomial, we are able to assure the existence of a set of coefficients depending on $\tau$, with $0\leqslant\tau\leqslant m-1$, $\{\hat{\xi}_\nu^{\,k+m}(\tau)\}_{k-m\leqslant\tau\leqslant k+m}$, with $\hat{\xi}_{k-m}^{\,k+m}(\tau)\neq 0$, realising the equality

$$\big(D^{k+\tau}P_{k+m}\big)(x)\sum_{\nu=k-\tau}^{k+\tau}\xi_\nu^{k+\tau}\,P_\nu(x) = \sum_{\nu=k-m}^{k+m}\hat{\xi}_\nu^{\,k+m}(\tau)\;P_\nu(x),\quad 0\leqslant\tau\leqslant m-1\,.$$

Therefore, (2.4.32) may be represented by

$$\begin{aligned}&\binom{k}{m}(k+m)!D^{k-m}\big(\lambda_0^k\vartheta_k\Phi^k(x)\,u_0\big)\\&=\left\{\Xi_m(k)P_{k+m}-\sum_{\tau=0}^{m-1}\binom{k}{\tau}\sum_{\nu=k-m}^{k+m}\hat{\xi}_\nu^{\,k+m}(\tau)\;P_\nu(x)\right\}u_0\,,\end{aligned}$$

consequently the equality (2.4.32) becomes

$$D^{k-m}\big(\lambda_0^k \vartheta_k \Phi^k(x)\, u_0\big) = S_{k+m}(x)\, u_0\ ,$$

with

$$\begin{aligned} S_{k+m}(x) &= \frac{1}{\binom{k}{m}(k+m)!}\Big\{\Xi_m(k)P_{k+m} - \sum_{\tau=0}^{m-1}\binom{k}{\tau}\sum_{\nu=k-m}^{k+m}\hat{\xi}_{\nu}^{\,k+m}(\tau)\ P_\nu(x)\Big\} \\ &= \frac{1}{\binom{k}{m}(k+m)!}\Big\{\Xi_m(k)P_{k+m} - \sum_{\nu=k-m}^{k+m}\Big(\sum_{\tau=0}^{m-1}\binom{k}{\tau}\hat{\xi}_{\nu}^{\,k+m}(\tau)\Big)\ P_\nu(x)\Big\}\ . \end{aligned}$$

Thus, the polynomial $S_{k+m}(x)$ may be neatly represented as

$$S_{k+m}(x) = \sum_{\nu=k-m}^{k+m} \zeta_\nu^{k+m} P_\nu(x)$$

if we consider

$$\zeta_\nu^{k+m} = \frac{1}{\binom{k}{m}(k+m)!}\sum_{\tau=0}^{m-1}\binom{k}{\tau}\hat{\xi}_{\nu}^{\,k+m}(\tau)\ , \quad 0 \leqslant \nu \leqslant k+m-1$$

and

$$\zeta_{k+m}^{k+m} = \frac{1}{\binom{k}{m}(k+m)!}\Big(\Xi_m(k) + \sum_{\tau=0}^{m-1}\binom{k}{\tau}\hat{\xi}_{k+m}^{\,k+m}(\tau)\Big)\ .$$

As a result, the equality (2.4.30)-(2.4.31) holds for any $0 \leqslant \mu \leqslant m \leqslant k$. The insertion of (2.4.30) in (2.4.26) provides

$$\sum_{\tau=0}^{k}\binom{k}{\tau} S_{k+\tau}(x)\ \big(D^{k+\tau}P_{n+k}\big)(x)\ u_0 = \Xi_n(k)\ P_{n+k}(x)\ u_0,$$

and, because of the regularity of $u_0$, permits to conclude that each element of $\{P_n\}_{n\in\mathbb{N}}$ is an eigenfunction of a differential equation of the type (2.4.1). Now, theorem 2.4.1 assures the classical character of $\{P_n\}_{n\in\mathbb{N}}$. □

Naturally, if we take $n = 0$ in (2.4.25), we recover the functional *Rodrigues type formula* (2.0.4) with $n$ replaced by $k$. On the other hand, taking $k = 1$ in (2.4.25) we meet the functional relation (2.0.6).

# CHAPTER 3

## Quadratic decomposition of some Appell sequences

Entailed in the problem of the symmetrysation of sequences of polynomials, comes out the quadratic decomposition(as well as the cubic decomposition) of a polynomial sequence. Within this context, many authors have dealt with symmetrization problems of orthogonal polynomial sequences either on the real line or in the unit circle. Among them we quote Barrucand and Dickinson [9], Chihara [24, 25, 26], L.M.Chihara and T.S.Chihara [27], Dickinson and Warsi [35], Geronimo and Van Assche [47], Maroni [80, 81]. In particular, in [26, 27] a symmetric orthogonal polynomial sequence is decomposed into two nonsymmetric sequences. As an example, we recall the well known relations linking the Hermite polynomials $\{H_n\}_{n\in\mathbb{N}}$ and the Laguerre polynomials $\{L_n(\cdot;\alpha)\}_{n\in\mathbb{N}}$, with $\alpha \neq -n$, $n \geqslant 1$ (*cf.* Carlitz [22] and also the brief paper of Al-Salam [3]):

$$H_{2n}(x) = L_n\left(x^2; -\frac{1}{2}\right) \quad , \quad H_{2n+1}(x) = x\, L_n\left(x^2; \frac{1}{2}\right), \quad n \in \mathbb{N}.$$

A generalisation of this idea came up with Maroni [80, 81] in the sense that for a given MPS $\{B_n\}_{n\in\mathbb{N}}$, we associate two other MPS, $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$, and two sequences of polynomials, $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, such that

$$B_{2n}(x) = P_n(x^2) + x\, a_{n-1}(x^2), \quad n \in \mathbb{N}, \tag{3.0.1}$$

$$B_{2n+1}(x) = b_n(x^2) + x\, R_n(x^2), \quad n \in \mathbb{N}. \tag{3.0.2}$$

where $\deg a_n \leqslant n$, $\deg b_n \leqslant n$, $n \in \mathbb{N}$, $a_{-1}(x) = 0$, (Maroni [80, 81]).

Any MPS $\{B_n\}_{n\in\mathbb{N}}$ may described by this approach, known as its **quadratic decomposition** (QD). It should be pointed out that under the assumption that $\{B_n\}_{n\in\mathbb{N}}$ is orthogonal, it is not possible to conclude that $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ are also orthogonal, if some supplementary conditions are not considered. For instance, $a_n = 0 = b_n$, $n \in \mathbb{N}$, if and only if the MPS $\{B_n\}_{n\in\mathbb{N}}$ is symmetric, that is $B_n(-x) = (-1)^n B_n(x), n \in \mathbb{N}$, and its orthogonality supplies the orthogonality of both sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ (Maroni [80]): it is indeed the case of the previously mentioned *Hermite* polynomials and the case of other symmetric sequences like the generalized *Hermite polynomials* (*cf.* Chihara's book [26, pp. 40-45]).

The quadratic decomposition of an MPS $\{B_n\}_{n\in\mathbb{N}}$ according to (3.0.1)-(3.0.2) is a particular case of a more general quadratic decomposition having as essential feature the fact that the argument of the four associated sequences is no longer $x^2$ but a two degree polynomial. Such general quadratic decomposition, expounded in the PhD thesis of A. Macedo [77], permits for example to quadratically decompose the elements of the symmetric *Gegenbauer* polynomial sequence (*cf.* [26, pp. 42]) among others in a more natural way. We will stop here the discussion of this more general QD as it will not be useful for the sequel.

Actually the sequence of *Hermite* polynomials is the most popular example of the so called *Appell* polynomial sequence (or, in short, *Appell sequences*) [8]. Inasmuch as *Appell sequences* are the cynosure of this chapter, we ought to define them formally.

**Definition 3.0.5** (Appell polynomial sequences [8])**.** A MPS $\{B_n\}_{n\in\mathbb{N}}$ is said to be an *Appell polynomial sequence* if the sequence of monic derivatives $\{B_n^{[1]}\}_{n\in\mathbb{N}}$ (defined in (1.3.6)) and the original one coincides, that is, $B_n(\cdot) = B_n^{[1]}(\cdot)$, $n \in \mathbb{N}$.

The notion of Appell polynomial sequences may be broadened to other linear and surjective operators, beside the differential operator $D$. Let us first clarify which type of linear operators are we interested in dealing with. The main focus is on the so called **lowering operator** which happen to be any linear surjective operator decreasing in one unit the degree of a polynomial. More formally, an operator $\mathcal{L}$ is said to be a lowering operator whenever it is linear, surjective ($\mathcal{L}(\mathcal{P}) = \mathcal{P}$) with $\mathcal{L}(1) = 0$ and $\deg\big(\mathcal{L}(x^n)\big) = n-1$, $n \in \mathbb{N}^*$. Obviously, $D$ satisfy such conditions.

It is possible to introduce lowering operators reducing the degree of a polynomial by $k \geqslant 2$ units, however, this is out of our interest for the moment.

Given a MPS $\{B_n\}_{n\in\mathbb{N}}$, we may construct a sequence of polynomials $\{B_n^{[1]}(\cdot, \mathcal{L})\}$ defined according to

$$B_n^{[1]}(x,\mathcal{L}) := \big(\rho_{n+1}(\mathcal{L})\big)^{-1}\,\mathcal{L}\left(B_{n+1}(x)\right)\ , \quad n\in\mathbb{N}. \tag{3.0.3}$$

where $\rho_{n+1}(\mathcal{L})\in\mathbb{C}-\{0\}$, $n\in\mathbb{N}$, is chosen so that $\mathcal{L}(x^{n+1})=\rho_{n+1}(\mathcal{L})\,x^n+q(x)$, for any $n\in\mathbb{N}$, and for some $q\in\mathcal{P}$ with $0\leqslant\deg q\leqslant n-1$. The new sequence $\{B_n^{[1]}(\cdot,\mathcal{L})\}_{n\in\mathbb{N}}$ is therefore a MPS.

**Definition 3.0.6** ($\mathcal{L}$-Appell polynomial sequences [15, 16]). A MPS $\{B_n\}_{n\in\mathbb{N}}$ is called an $\mathcal{L}$-Appell sequence with respect to a lowering operator $\mathcal{L}$ if $B_n(\cdot)=B_n^{[1]}(\cdot,\mathcal{L})$, $n\in\mathbb{N}$, with $B_n^{[1]}(\cdot,\mathcal{L})$ defined according to (3.0.3).

Based upon the previous definitions, the Appell sequences are the $D$-Appell sequences. Unless the context requires more precision, we will keep the first terminology.

During this decade we have witnessed to an increasing interest about the Appell sequences with respect to lowering operators, the now called $\mathcal{L}$-Appell sequences. Particularly, Ben Cheikh [15, 16] has expounded this matter by giving a more concise interpretation. Besides, several authors have also given a special attention to such sequences, among them we quote the works of Ben Cheikh and Gaied [13], Cesarano [23], Dattoli [31], Dattoli et al. [32], Ghressi and Khérigi [49], Ghressi and Khériji [50, 51], He and Ricci [55] (see also Ismail [58]), Maroni and Mejri [91] and Srivastava [101].

As examples of lowering operators considered are the $q$-derivative $H_q$, which will be in focus in section 3.8 (Ghressi and Khériji [51], Ismail [57], Khériji and Maroni [59], Maroni [84]); the Hahn's operator $D_\omega$ of finite differences (Abdelkarim and Maroni [1], Maroni [84]) with $\Big(D_\omega f\Big)(x):=\frac{f(x+\omega)-f(x)}{\omega}$, for $f\in\mathcal{P}$ and $\omega\in\mathbb{C}^*$; the *Dunkl operator* $\mathcal{D}_\theta:=D+\theta H_{-1}$ for $\theta\in\mathbb{C}^*$ introduced by Dunkl [38] (Ben Cheikh and Gaied [12, 13], Ghressi and Khériji [50]); the differential operators like $DxD$ or more general $\sum_{\nu=0}^{k}a_\nu x^\nu\,D^{\nu+1}$ with $k\in\mathbb{N}$ and $a_\nu\in\mathbb{C}$ with $\prod_\nu a_\nu\neq 0$, (Ben Cheikh [15, 16], Blasiak et al. [17], Dattoli [31], Dattoli et al. [32, 33, 34]).

Within this context, we intend to give a small contribution to the theory. The forthcoming developments are mainly concerned with the quadratic decomposition of an Appell sequence. The four associated sequences to this QD happen to be also Appell sequences but with respect to another lowering operator, which we have called $\mathcal{F}_\varepsilon$. Therefore in section 3.2 a description from a functional point of view (*i.e.* based on the theory of linear functionals) of all the $\mathcal{F}_\varepsilon$-Appell sequences will be given. Nevertheless, the highest attraction among these sequences resides in those possessing orthogonality, which are essentially the Laguerre sequences of parameter $\varepsilon/2$, up to a linear change of variable (see theorem 3.3.1). Later on, in section 3.4 we will repeat this same procedure but with the $\mathcal{F}_\varepsilon$-Appell sequences playing the role of the $(D)$-Appell sequences. Again, we show that the four associated sequences to the QD of an $\mathcal{F}_\varepsilon$-Appell sequence are Appell sequences with respect to another lowering operator, here denoted as $\mathcal{G}_{\varepsilon,\mu}$ (see theorem 3.4.1). After characterising the arising $\mathcal{G}_{\varepsilon,\mu}$-Appell sequences, we realise the impossibility of any of such sequences to be (regularly) orthogonal. In spite of this negative result, based on theorems 3.4.1 and 3.4.1, we are able to, in section 3.7, completely describe the QD of a Laguerre sequence. As previously announced, we are also interested in exploring the $q$-Appell character of a sequence (that is, Appell sequences with respect to the *$q$-derivative operator*) through this approach and consistent results are obtained. This occurs in section 3.8.

## 3.1 Quadratic decomposition of an Appell sequence

Proceeding to the QD of an Appell sequence $\{B_n\}_{n\in\mathbb{N}}$ in accordance with (3.0.1)-(3.0.2), we are ready to characterise the two associated MPS $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$, as well as the two polynomial sets $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$.

**Theorem 3.1.1.** *[72] Consider the quadratic decomposition of a monic sequence $\{B_n\}_{n\in\mathbb{N}}$ as in (3.0.1)-(3.0.2). If $\{B_n\}_{n\in\mathbb{N}}$ is an Appell sequence, then the four associated sequences $\{P_n\}_{n\in\mathbb{N}}$, $\{R_n\}_{n\in\mathbb{N}}$, $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are given by*

$$P_n(x) = \frac{1}{(n+1)(2n+1)}\big(\mathcal{F}_{-1}P_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.1.1}$$

$$R_n(x) = \frac{1}{(n+1)(2n+3)}\big(\mathcal{F}_{1}R_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.1.2}$$

$$a_n(x) = \frac{1}{(n+2)(2n+3)}\big(\mathcal{F}_{1}a_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.1.3}$$

$$b_n(x) = \frac{1}{(n+1)(2n+3)}\big(\mathcal{F}_{-1}b_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.1.4}$$

*where the operator $\mathcal{F}_\varepsilon$ (with $\varepsilon=1$ or $\varepsilon=-1$) is given by*

$$\mathcal{F}_\varepsilon = 2DxD + \varepsilon D \quad \textit{with} \quad D = \frac{d}{dx}\,. \tag{3.1.5}$$

*Proof.* Indeed, by differentiating (3.0.1) and (3.0.2) with $n$ replaced by $n+1$, then, under the assumption, we obtain:

$$(2n+2)\{b_n(x^2) + x\,R_n(x^2)\} = 2(n+1)x\,P_n^{[1]}(x^2) + a_n(x^2) + 2x^2a_n'(x^2), n\in\mathbb{N},$$
$$(2n+1)\{P_n(x^2) + x\,a_{n-1}(x^2)\} = 2x\,b_n'(x^2) + R_n(x^2) + 2n\,x^2\,R_{n-1}^{[1]}(x^2), n\in\mathbb{N},$$

which consists of polynomials with only even or odd powers. As a result, we necessarily get:

$$P_n^{[1]}(x) = R_n(x),\ n \in \mathbb{N}, \tag{3.1.6}$$

$$(2n+1)P_n(x) = R_n(x) + 2n\,x\,R_{n-1}^{[1]}(x),\ n \in \mathbb{N}, \tag{3.1.7}$$

$$(2n+2)b_n(x) = a_n(x) + 2xa_n'(x),\ n \in \mathbb{N}, \tag{3.1.8}$$

$$(2n+1)a_{n-1}(x) = 2\,b_n'(x),\ n \in \mathbb{N}. \tag{3.1.9}$$

In (3.1.7), making $n \to n+1$, by differentiating on both sides and using (3.1.6), we obtain

$$(n+1)(2n+3)R_n(x) = 2\left(x\,R_{n+1}'(x)\right)' + R_{n+1}'(x),\ n \in \mathbb{N}. \tag{3.1.10}$$

On the other hand, we may express (3.1.7) only in terms of elements of $\{P_n\}_{n\in\mathbb{N}}$ and its derivatives, by taking into consideration (3.1.6). Thus, we get:

$$(n+1)(2n+1)P_n(x) = 2\left(x\,P_{n+1}'(x)\right)' - P_{n+1}'(x),\ n \in \mathbb{N}. \tag{3.1.11}$$

Hence, the relations (3.1.10) and (3.1.11) may be respectively expressed as follows:

$$R_n(x) = \frac{1}{(n+1)(2n+3)}\big(2DxD + D\big)R_{n+1}(x),\ n \in \mathbb{N}, \tag{3.1.12}$$

and

$$P_n(x) = \frac{1}{(n+1)(2n+1)}\big(2DxD - D\big)P_{n+1}(x),\ n \in \mathbb{N}. \tag{3.1.13}$$

In addition, we may express (3.1.8) exclusively in terms depending on $b_n$ and its derivatives by taking into account (3.1.9). In a simplified way, we obtain

$$b_n(x) = \frac{1}{(n+1)(2n+3)}\big(2DxD - D\big)b_{n+1}(x),\ n \in \mathbb{N}. \tag{3.1.14}$$

From (3.1.9) and on account of (3.1.8), we get

$$a_n(x) = \frac{1}{(n+2)(2n+3)}\big(2DxD + D\big)a_{n+1}(x),\ n \in \mathbb{N}. \tag{3.1.15}$$

□

The information about the sets of polynomials $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ may be improved, as it is shown in the very next result.

**Proposition 3.1.2.** *[72] Let $\{B_n\}_{n\in\mathbb{N}}$ be an Appell sequence and let (3.0.1)-(3.0.2) be its quadratic decomposition. Then, either $\{B_n\}_{n\in\mathbb{N}}$ is symmetric or there exists an integer $p \geqslant 0$*

*such that $a_p(\cdot) \neq 0$ (respectively, $b_p(\cdot) \neq 0$). In this case, we have*

$$a_n(x) = 0,\ b_n(x) = 0,\ 0 \leqslant n \leqslant p-1,\ \textit{when}\ p \geqslant 1, \tag{3.1.16}$$

$$a_{p+n}(x) = \binom{n+p+1}{n} \frac{\left(p+\frac{3}{2}\right)_n}{\left(\frac{3}{2}\right)_n}\, a_p\ \widehat{a}_n(x)\,, \quad n \in \mathbb{N}, \tag{3.1.17}$$

$$b_{p+n}(x) = \binom{n+p}{n} \frac{\left(p+\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_n}\, b_p\ \widehat{b}_n(x)\,, \quad n \in \mathbb{N} \tag{3.1.18}$$

*where $\widehat{a}_n$ and $\widehat{b}_n$ are two monic polynomials fulfilling $\deg \widehat{a}_n(x) = n$ and $\deg \widehat{b}_n(x) = n,\ n \in \mathbb{N}$ and the $(a)_n = a\,(a+1)\ldots(a+n-1)$ represents the Pochhammer symbol.*

*Proof.* If $\{B_n\}_{n\in\mathbb{N}}$ is a symmetric sequence, then $a_n(\cdot) = 0,\ n \in \mathbb{N}$, and also $b_n(\cdot) = 0,\ n \in \mathbb{N}$. Reciprocally, if $a_n(\cdot) = 0,\ n \in \mathbb{N}$ (respectively, $b_n(\cdot) = 0,\ n \in \mathbb{N}$), then from (3.1.8) $b_n(\cdot) = 0,\ n \in \mathbb{N}$ (respectively $a_n(\cdot) = 0,\ n \in \mathbb{N}$, from (3.1.9) ).
When $\{B_n\}_{n\in\mathbb{N}}$ is not a symmetric sequence, let $p \geqslant 0$ be the smallest integer such that $a_p(\cdot) \neq 0$ and $a_n(\cdot) = 0,\ 0 \leqslant n \leqslant p-1$ when $p \geqslant 1$ . From (3.1.9), we have $b_n(\cdot) =$ constant, $0 \leqslant n \leqslant p$ and by virtue of (3.1.8), $b_n(\cdot) = 0$ for $0 \leqslant n \leqslant p-1$, $(2p+2)b_p = a_p(x) + 2x a_p'(x)$, which implies $a_p(\cdot) =$ constant $= a_p \neq 0$. Thus, $a_p = (2p+2)b_p$.
Proceeding by finite induction, by taking into account (3.1.8)-(3.1.9), we achieve the conclusion that $\deg(a_{n+p}) = n$ and $\deg(b_{n+p}) = n,\ n \in \mathbb{N}$. Therefore we may consider two nonzero sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} a_{n+p}(x) &= \lambda_n\ \widehat{a}_n(x)\ , \\ b_{n+p}(x) &= \mu_n\ \widehat{b}_n(x)\ , n \in \mathbb{N}, \end{aligned}$$

where $\widehat{a}_n(\cdot)$ and $\widehat{b}_n(\cdot)$ are two monic polynomials of degree $n,\ n \in \mathbb{N}$, $\mu_0 = b_p$ and $\lambda_0 = 2(p+1)\,b_p$ . Due to (3.1.8)-(3.1.9) we deduce that

$$\lambda_n = \binom{n+p+1}{n} \frac{\left(p+\frac{3}{2}\right)_n}{\left(\frac{3}{2}\right)_n}\, \lambda_0,$$

$$\mu_n = \frac{n+\frac{1}{2}}{n+p+1}\, \lambda_n\ , \quad n \in \mathbb{N},$$

whence the result. □

Just like the differential operator $D$, the operator $\mathcal{F}_\varepsilon$ given by (3.1.5) is a lowering operator decreasing in one unit the degree of a polynomial with $\mathcal{F}_\varepsilon(1) = 0$. Given a MPS $\{B_n\}_{n\in\mathbb{N}}$, we define a MPS $\{B_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ through

$$B_n^{[1]}(x;\mathcal{F}_\varepsilon) = \frac{1}{(n+1)\big(2(n+1)+\varepsilon\big)}\,\mathcal{F}_\varepsilon\Big(B_{n+1}(x)\Big), \quad n\in\mathbb{N}. \tag{3.1.19}$$

According to (3.1.19), we may read from theorem 3.4.1 that the two MPS $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ associated to the quadratic decomposition of the $D$-Appell sequence $\{B_n\}_{n\in\mathbb{N}}$, are such that

$$P_n(x) = P_n^{[1]}(x;\mathcal{F}_{-1}), \quad n\in\mathbb{N},$$
$$R_n(x) = R_n^{[1]}(x;\mathcal{F}_{1}), \quad n\in\mathbb{N}\,.$$

Therefore, based upon definition 3.0.6, the polynomial sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ are $\mathcal{F}_{-1}$-Appell and $\mathcal{F}_1$-Appell, respectively. Likewise, by virtue of the relations (3.1.3)-(3.1.4) together with (3.1.17)-(3.1.18), the sequences $\{\widehat{a}_n\}_{n\in\mathbb{N}}$ and $\{\widehat{b}_n\}_{n\in\mathbb{N}}$ arisen from proposition 3.1.2 are $\mathcal{F}_1$ and $\mathcal{F}_{-1}$-Appell, respectively.

In view of a more complete description about the four associated sequences to the quadratic decomposition of an Appell sequence, the characterisation of all the $\mathcal{F}_\varepsilon$-Appell sequences is now on target, and will be carried out in the next section. Insofar as there is no reason to confine the study of the $\mathcal{F}_\varepsilon$-Appell sequences whether $\varepsilon$ is $1$ or $(-1)$, we will broaden the range of the parameter $\varepsilon$ to the set of all complex numbers excluding the negative even integers. In other words, from now on we will be considering $\varepsilon$ to be such that

$$\varepsilon\in\mathbb{C}\backslash\big\{-2n,\ n\in\mathbb{N}^*\big\} \tag{3.1.20}$$

## 3.2 The $\mathcal{F}_\varepsilon$-Appell sequences

Consider $\{B_n\}_{n\in\mathbb{N}}$ to be a MPS and $\{u_n\}_{n\in\mathbb{N}}$ its corresponding dual sequence. Let us denote by $\{u_n^{[1]}(\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ the dual sequence associated to the MPS $\{B_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ given by (3.1.19).

Aware of the relation between the elements of $\{B_n\}_{n\in\mathbb{N}}$ and those of $\{B_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$, we now intend to find the relation between the elements of the corresponding corresponding dual sequences. Regarding this purpose, we shall first figure out the transposed of the operator $\mathcal{F}_\varepsilon$, denoted here as ${}^t\mathcal{F}_\varepsilon$. Following (1.1.2) and(1.1.4), by duality we successively have

$$\begin{aligned}\langle {}^t\mathcal{F}_\varepsilon u\,,\,f\rangle &= \langle u\,,\,\mathcal{F}_\varepsilon f\rangle = \langle u\,,\,\big(2\,D\,x\,D+\varepsilon\,D\big)f\rangle\\ &= \langle \big(2\,D\,x\,D-\varepsilon\,D\big)\,u\,,\,f\rangle\,,\end{aligned}$$

therefore ${}^t\mathcal{F}_\varepsilon = \big(2\,D\,x\,D - \varepsilon\,D\big)$. However the convention on the differential operator $D$ (${}^tD = -D$) permits to write ${}^t\mathcal{F}_\varepsilon = 2\,D\,x\,D-\varepsilon\,D$, leaving out a slight abuse of notation without consequence. Thus ${}^t\mathcal{F}_\varepsilon := \mathcal{F}_{-\varepsilon}$ and $\mathcal{F}_\varepsilon$ is defined either on $\mathcal{P}$ and $\mathcal{P}'$, and the following easy to prove properties are valid

$$\mathcal{F}_\varepsilon(pf) = f(\mathcal{F}_\varepsilon p) + p(\mathcal{F}_\varepsilon f) + 4xp'f'\,, \quad p, f \in \mathcal{P}, \tag{3.2.1}$$

$$\mathcal{F}_{-\varepsilon}(p\,u) = p\,\mathcal{F}_{-\varepsilon}(u) - \mathcal{F}_\varepsilon(p)\,u\; + 4\big(x\,p'\,u\big)', \quad p \in \mathcal{P},\, u \in \mathcal{P}'\,. \tag{3.2.2}$$

**Lemma 3.2.1.** *[72] The dual sequence* $\{u_n^{[1]}(\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ *fulfils*

$$\mathcal{F}_{-\varepsilon}\big(u_n^{[1]}(\mathcal{F}_\varepsilon)\big) = (n+1)\big(2(n+1)+\varepsilon\big)u_{n+1}, \quad n \in \mathbb{N}. \tag{3.2.3}$$

*Proof.* Indeed, successively we have

$$\begin{aligned}
&\langle u_n^{[1]}(\mathcal{F}_\varepsilon), B_m^{[1]}(x;\mathcal{F}_\varepsilon)\rangle = \delta_{n,m}\,, \quad n, m \geqslant 0,\\
&\langle u_n^{[1]}(\mathcal{F}_\varepsilon), \mathcal{F}_\varepsilon(B_{m+1})\rangle = (n+1)\big(2(n+1)+\varepsilon\big)\,\delta_{n,m}\,, \quad n, m \geqslant 0,\\
&\langle \mathcal{F}_{-\varepsilon}\big(u_n^{[1]}(\mathcal{F}_\varepsilon)\big), B_{m+1}\rangle = (n+1)\big(2(n+1)+\varepsilon\big)\,\delta_{n,m}\,, \quad n, m \geqslant 0.
\end{aligned} \tag{3.2.4}$$

In particular,

$$\langle \mathcal{F}_{-\varepsilon}\big(u_n^{[1]}(\mathcal{F}_\varepsilon)\big), B_{m+1}\rangle = 0, \quad m \geqslant n+1,\, n \in \mathbb{N}.$$

On account of lemma 1.3.1, this implies

$$\mathcal{F}_{-\varepsilon}\big(u_n^{[1]}(\mathcal{F}_\varepsilon)\big) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu}\, u_\nu, \quad n \in \mathbb{N},$$

with $\lambda_{n,\nu} = \langle \mathcal{F}_{-\varepsilon}\big(u_n^{[1]}(\mathcal{F}_\varepsilon)\big), B_\nu\rangle$, $0 \leqslant \nu \leqslant n+1$. As a consequence, on account of (3.2.4), we obtain (3.2.3). □

Now the attention returns to our primary purpose of describing the dual sequence of a $\mathcal{F}_\varepsilon$-Appell sequence. Based on the previous result, we obtain the following one:

**Proposition 3.2.2.** *[72] The MPS* $\{B_n\}_{n\in\mathbb{N}}$ *is a* $\mathcal{F}_\varepsilon$*-Appell sequence if and only if its dual sequence* $\{u_n\}_{n\in\mathbb{N}}$ *fulfils*

$$u_n = \frac{1}{n!\,2^n\,\left(1+\frac{\varepsilon}{2}\right)_n}\,\mathcal{F}_{-\varepsilon}^{\,n}\,(u_0), \quad n \in \mathbb{N}\,. \tag{3.2.5}$$

*Proof.* The condition is necessary. From (3.2.3), the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies

$$\mathcal{F}_{-\varepsilon}(u_n) = (n+1)\big(2(n+1)+\varepsilon\big)u_{n+1}, \quad n \in \mathbb{N}. \tag{3.2.6}$$

In particular, for $n = 0$,

$$u_1 = \frac{1}{2+\varepsilon}\,\mathcal{F}_{-\varepsilon}\,u_0\,.$$

By recurrence, we get (3.2.5).

The condition is sufficient. From (3.2.5), it is easy to see that (3.2.6) is fulfilled. Therefore by comparing it with (3.2.3), we obtain

$$\mathcal{F}_{-\varepsilon}\left(u_n^{[1]}(\mathcal{F}_\varepsilon)\right) = \mathcal{F}_{-\varepsilon}\, u_n,\ n \in \mathbb{N}\,.$$

The lowering operator $\mathcal{F}_\varepsilon$ satisfies $\mathcal{F}_\varepsilon(\mathcal{P}) = \mathcal{P}$, and therefore $\mathcal{F}_{-\varepsilon}$ is one-to-one on $\mathcal{P}'$. We then get $u_n^{[1]}(\mathcal{F}_\varepsilon) = u_n$ , $n \in \mathbb{N}$, whence the expected result. □

Among all the $\mathcal{F}_\varepsilon$-Appell sequences we are particularly interested in ferreting out the orthogonal ones. As a matter of fact, up to a linear change of variable, the Hermite polynomials form the only sequence of polynomials that are simultaneously $D$-Appell and orthogonal. Such characterisation was first given by Angelescu [7] and later by other authors (see, *e.g.*, Shohat [98], Rainville [94, p.187] and for further references consult Al-Salam [4]).

## 3.3 The $\mathcal{F}_\varepsilon$-Appell orthogonal sequences

As previously pointed out in Chapter 2, the elements of a classical polynomial sequence are eigenfunctions of a second order differential equation (the so called *Bochner* differential equation) given by (2) and they also fulfil the "structural relation" (2.0.5). Particularly, according to Table 2.1 the elements of the canonical Laguerre polynomial sequence $\{P_n(\cdot;\alpha)\}_{n\in\mathbb{N}}$, with $\alpha \neq -\frac{n}{2}$, fulfil

$$\begin{aligned} x\,P''_{n+1}(x) - (x-\alpha-1)P'_{n+1}(x) &= -(n+1)P_{n+1}(x)\ , \quad n \in \mathbb{N}, \\ x\,P'_{n+1}(x) &= (n+1)P_{n+1}(x) + (n+1)(n+1+\alpha)P_n(x)\ , \quad n \in \mathbb{N}\,. \end{aligned}$$

Between the two equations we proceed to the elimination of the term in $x\,P'_{n+1}(x)$, and this provides

$$(n+1)(n+1+\alpha)P_n(x) = x\,P''_{n+1}(x) + (\alpha+1)P'_{n+1}(x)\ , \quad n \in \mathbb{N},$$

*i.e.* ,

$$(n+1)\big(2(n+1)+2\alpha\big)P_n(x) = \mathcal{F}_{2\alpha}\Big(P_{n+1}(x)\Big)\ , \quad n \in \mathbb{N},$$

which brings into light the fact the Laguerre polynomial sequence with parameter $\alpha$ is $\mathcal{F}_{2\alpha}$-Appell sequence and, of course, also orthogonal. (Notice that this conclusion was also achieved by Ben Cheikh and Srivastava [14], p. 423). Hitherto, the existence of orthogonal $\mathcal{F}_\varepsilon$-Appell sequence is assured. Nevertheless we intend to know whether there are any other than the Laguerre polynomial sequences of parameter $\varepsilon/2$. The next result brings the answer.

**Theorem 3.3.1.** *[72] All the $\mathcal{F}_\varepsilon$-Appell orthogonal sequences are the Laguerre polynomial sequences with parameter $\alpha = \frac{\varepsilon}{2}$, up to an affine transformation.*

*Proof.* Assume that the MOPS $\{B_n\}_{n\in\mathbb{N}}$ is also a $\mathcal{F}_\varepsilon$-Appell sequence. Consider $(\beta_n\, ,\, \gamma_{n+1})_{n\in\mathbb{N}}$ to be the recurrence coefficients of the second order recurrence relation fulfilled by the MOPS $\{B_n\}_{n\in\mathbb{N}}$. In addition, the corresponding dual sequence satisfies (1.4.2), which combined with (3.2.6), permits to conclude

$$\mathcal{F}_{-\varepsilon}(B_n u_0) = \lambda_n\, B_{n+1} u_0, \quad n \in \mathbb{N}, \tag{3.3.1}$$

with

$$\lambda_n := \lambda_n(\varepsilon) = \frac{(n+1)\big(2(n+1)+\varepsilon\big)}{\gamma_{n+1}}, \quad n \in \mathbb{N}, \tag{3.3.2}$$

Remark that $\lambda_n \neq 0$, $n \in \mathbb{N}$, since $\varepsilon \neq -2(n+1), n \in \mathbb{N}$. When we consider $n = 0$ in (3.3.1), we get

$$\mathcal{F}_{-\varepsilon}\, u_0 = \lambda_0\, B_1 u_0 \tag{3.3.3}$$

which is equivalent to

$$2(x\, u_0')' - \varepsilon u_0' = \lambda_0\, B_1 u_0. \tag{3.3.4}$$

On account of (3.2.2) and (3.3.3), from the relation (3.3.1) with $n = 1$ we deduce

$$4x\, u_0' = A(x)\, u_0\, , \tag{3.3.5}$$

where

$$A(x) = \lambda_1\, B_2(x) - \lambda_0\, B_1^2(x) - 2 + \varepsilon. \tag{3.3.6}$$

Differentiating both sides of (3.3.5) and using (3.3.4), we obtain

$$\big(A(x) - 2\varepsilon\big)\, u_0' = \big(2\lambda_0 B_1(x) - A'(x)\big)\, u_0\, .$$

Between (3.3.5) and this last equation we eliminate $u_0'$. Consequently, based on the regularity of $u_0$, it emerges the condition

$$\big(A(x) - 2\varepsilon\big) A(x) = 4x\, \big(2\lambda_0 B_1(x) - A'(x)\big)\, . \tag{3.3.7}$$

On the strength of (3.3.6) and (3.3.7), it is easily seen that $\lambda_1 = \lambda_0$, which implies

$$\begin{cases} \lambda_0(\beta_0 - \beta_1)^2 = 8 \\ 4\beta_0 + \lambda_0\gamma_1(\beta_0 - \beta_1) = 0 \\ \big(\lambda_0\gamma_1 + 2 + \varepsilon\big)\big(\lambda_0\gamma_1 + 2 - \varepsilon\big) + 4\lambda_0\beta_0\big(\beta_0 - \beta_1\big) = 0\, . \end{cases}$$

Nonetheless, in view of (3.3.2) with $n=0$, $\lambda_0\,\gamma_1 = 2+\varepsilon$ , whence

$$\beta_1 = \left(1+\frac{4}{2+\varepsilon}\right)\beta_0 \qquad , \qquad \beta_0 = \sqrt{\frac{2}{\lambda_0}}\,\left(1+\frac{\varepsilon}{2}\right)$$

and $A(x) = -2\sqrt{2\lambda_0}\;x + 2\,\varepsilon\,,$ where the last equalities are obtained up to a reflection. Following (3.3.5), we deduce that $u_0$ fulfils the functional differential equation

$$\Big(\Phi\, u_0\Big)' + \Psi\, u_0 = 0 \tag{3.3.8}$$

with $\Phi(x) = x$ and $\Psi(x) = \sqrt{\dfrac{\lambda_0}{2}}\;x - \left(1+\dfrac{\varepsilon}{2}\right)$ . Therefore, according to (2.0.1), $u_0$ is a classical form. The information given by Table 2.1, permits to conclude that (3.3.8) essentially corresponds to the functional equation of a Laguerre form with $\alpha = \dfrac{\varepsilon}{2}$ and up to the affine transformation $\sqrt{\dfrac{\lambda_0}{2}}\,x$. □

**Remark 3.3.1.** With the following definition "*a MOPS $\{B_n\}_{n\in\mathbb{N}}$ is called a $\mathcal{F}_\varepsilon$-classical sequence when $\{B_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ is also orthogonal (Hahn property with respect to $\mathcal{F}_\varepsilon$)*", the monic Laguerre sequence with parameter $\dfrac{\varepsilon}{2}$ is a $\mathcal{F}_\varepsilon$-classical sequence since $B_n^{[1]}(x;\mathcal{F}_\varepsilon) = B_n(x)$, $n\in\mathbb{N}$, and the Laguerre form $u_0$ fulfilling (3.3.3) is a $\mathcal{F}_\varepsilon$-classical form. It is well known that the monic Hermite sequence possesses the same properties with respect to the operator $D$ [7]. Hence, this compels us to approach the study of all the $\mathcal{F}_\varepsilon$-classical sequences, which will be the main target of Chapter 4.

## 3.4 Quadratic decomposition of an Appell sequence with respect to a second order differential lowering operator

Pursuing the idea of the QD of an Appell sequence, we now explore the $\mathcal{F}_\varepsilon$-Appell sequences. Regarding this issue, it is useful to focus on some properties of the operator $\mathcal{F}_\varepsilon$; namely for any $f,g\in\mathcal{P}$, we have:

$$\begin{aligned}
\mathcal{F}_\varepsilon\big(f(x)\,g(x)\big) &= f(x)\,\mathcal{F}_\varepsilon\big(g(x)\big) + g(x)\,\mathcal{F}_\varepsilon\big(f(x)\big) + 4\,x\,f'(x)\,g'(x),\\
\mathcal{F}_\varepsilon\big(\,f(t^2)\,\big)(x) &= x\,\Big\{8\,x^2\,f''(x^2) + 2(4+\varepsilon)\,f'(x^2)\Big\}\,, && (3.4.1)\\
\mathcal{F}_\varepsilon\big(t\,f(t^2)\big)(x) &= x^2\,\Big\{8\,x^2\,f''(x^2) + 2(8+\varepsilon)\,f'(x^2)\Big\} + (2+\varepsilon)\,f(x^2)\,. && (3.4.2)
\end{aligned}$$

The relations (3.4.1)-(3.4.2) may be equivalently written like

$$\begin{aligned}
&\mathcal{F}_\varepsilon\big(f(t^2)\big)(x) = 4x\big(\mathcal{F}_{\varepsilon/2}\,f(t)\big)(x^2)\\
&\mathcal{F}_\varepsilon\big(t\,f(t^2)\big)(x) = 4x^2\big(\mathcal{F}_{\varepsilon/2}\,f(t)\big)(x^2) + 8x^2 f'(x^2) + (2+\varepsilon)f(x^2)
\end{aligned}$$

**Theorem 3.4.1.** *Consider the quadratic decomposition of a monic sequence* $\{B_n\}_{n\in\mathbb{N}}$ *as in (3.0.1)-(3.0.2). If* $\{B_n\}_{n\in\mathbb{N}}$ *is an* $\mathcal{F}_\varepsilon$*-Appell sequence with* $\varepsilon \neq -2(n+1)$, $n \in \mathbb{N}$, *then the four associated sequences* $\{P_n\}_{n\in\mathbb{N}}$, $\{R_n\}_{n\in\mathbb{N}}$, $\{a_n\}_{n\in\mathbb{N}}$ *and* $\{b_n\}_{n\in\mathbb{N}}$ *are given by*

$$P_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,-1)}\,\big(\mathcal{G}_{\varepsilon,-1} P_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.4.3}$$

$$R_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,1)}\,\big(\mathcal{G}_{\varepsilon,1} R_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.4.4}$$

$$a_n(x) = \frac{1}{\eta_{n+2}(\varepsilon,-1)}\,\big(\mathcal{G}_{\varepsilon,1} a_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.4.5}$$

$$b_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,1)}\,\big(\mathcal{G}_{\varepsilon,-1} b_{n+1}\big)(x),\ n\in\mathbb{N}, \tag{3.4.6}$$

*where*

$$\mathcal{G}_{\varepsilon,1} = \big(4DxD+\varepsilon D\big)\ \big(2xD+\mathrm{I}_{\mathcal{P}}\big)\ \big(4xD+(2+\varepsilon)\mathrm{I}_{\mathcal{P}}\big) \tag{3.4.7}$$

$$\mathcal{G}_{\varepsilon,-1} = \big(4DxD+\varepsilon D\big)\ \big(2xD-\mathrm{I}_{\mathcal{P}}\big)\ \big(4xD-(2-\varepsilon)\mathrm{I}_{\mathcal{P}}\big) \tag{3.4.8}$$

*and*

$$\eta_{n+1}(\varepsilon,1) = (n+1)\ \big(4(n+1)+\varepsilon\big)\ \big(2n+3\big)\ \big[2\ \big(2\,n+3\big)+\varepsilon\big] \neq 0\ ,\ n\in\mathbb{N}, \tag{3.4.9}$$

$$\eta_{n+1}(\varepsilon,-1) = (n+1)\ \big(4(n+1)+\varepsilon\big)\big(2n+1\big)\big[2\ \big(2\,n+1\big)+\varepsilon\big] \neq 0,\ n\in\mathbb{N}, \tag{3.4.10}$$

*with* $D := \frac{d}{dx}$ *and* $\mathrm{I}_{\mathcal{P}}$ *representing the identity on* $\mathcal{P}$.

*Proof.* Consider $\rho_{n+1} = (n+1)\big(2(n+1)+\varepsilon\big)$. Operating with $\mathcal{F}_\varepsilon$ on both members of (3.0.1) and (3.0.2) with $n$ replaced by $n+1$, then, under the assumption and by virtue of (3.4.1)-(3.4.2), we obtain:

$$\begin{aligned}
\rho_{2n+2}\{b_n(x^2)+x\,R_n(x^2)\} \quad &= \quad x\,\{2(4+\varepsilon)\,P'_{n+1}(x^2)+8\,x^2\,P''_{n+1}(x^2)\}\\
&\quad +(2+\varepsilon)\,a_n(x^2)+2(8+\varepsilon)\,x^2\,a'_n(x^2)\\
&\quad +8\,x^4\,a''_n(x^2)\ ,\quad n\in\mathbb{N},\\
\rho_{2n+1}\{P_n(x^2)+x\,a_{n-1}(x^2)\} \quad &= \quad x\,\{2(4+\varepsilon)\,b'_n(x^2)+8\,x^2\,b''_n(x^2)\}\\
&\quad +(2+\varepsilon)\,R_n(x^2)+2(8+\varepsilon)\,x^2\,R'_n(x^2)\\
&\quad +8\,x^4\,R''_n(x^2)\ ,\quad n\in\mathbb{N},
\end{aligned}$$

which consists of polynomials with only even or odd powers. As a result, we necessarily get:

$$\rho_{2n+2}\ R_n(x) = \Big\{2(4+\varepsilon)\ D+8\,x\ D^2\Big\}\ \Big(P_{n+1}(x)\Big),\quad n\in\mathbb{N}, \tag{3.4.11}$$

$$\rho_{2n+1}\ P_n(x) = \Big\{(2+\varepsilon)\ \mathrm{I}_{\mathcal{P}}+2(8+\varepsilon)\ x\ D+8\,x^2\ D^2\Big\}\ \Big(R_n(x)\Big),\quad n\in\mathbb{N}, \tag{3.4.12}$$

$$\rho_{2n+2}\ b_n(x) = \Big\{(2+\varepsilon)\ \mathrm{I}_{\mathcal{P}}+2(8+\varepsilon)\ x\ D+8\,x^2\ D^2\Big\}\ \Big(a_n(x)\Big),\quad n\in\mathbb{N}, \tag{3.4.13}$$

$$\rho_{2n+1}\ a_{n-1}(x) = \Big\{2(4+\varepsilon)\ D+8\,x\ D^2\Big\}\ \Big(b_n(x)\Big),\quad n\in\mathbb{N}. \tag{3.4.14}$$

Operating with the equalities (3.4.11) and (3.4.12), we deduce that

$$\begin{aligned}&\rho_{2n+2}\rho_{2n+3}\ R_n(x)\\&=\Big\{2\ \varepsilon\ D+8\ D\ x\ D\Big\}\cdot\Big\{(2+\varepsilon)\ \mathrm{I}_{\mathcal{P}}+2(4+\varepsilon)\ x\ D+8\ x\ D\ x\ D\Big\}\ \Big(R_{n+1}(x)\Big),\end{aligned}$$

holds for any $n\in\mathbb{N}$, and it is also valid

$$\begin{aligned}&\rho_{2n+1}\rho_{2n+2}\ P_n(x)\\&=\Big\{(2+\varepsilon)\ \mathrm{I}_{\mathcal{P}}+2(8+\varepsilon)\ x\ D+8\ x^2\ D^2\Big\}\cdot\Big\{2(4+\varepsilon)\ D+8\ x\ D^2\Big\}\ \Big(P_{n+1}(x)\Big),\end{aligned}$$

for any $n\in\mathbb{N}$. Using the identities

$$\left\{\begin{array}{l}x\ D^2=D\ x\ D-D\\ x^2\ D^2=x\ D\ x\ D-x\ D\end{array}\right.\quad\text{and}\quad\left\{\begin{array}{l}D\ x=x\ D-\mathrm{I}_{\mathcal{P}}\\ x^2\ D^2=D\ x\ D\ x-3\ D\ x+2\mathrm{I}_{\mathcal{P}}\end{array}\right.\tag{3.4.15}$$

in the right-hand side of the first and second previous relations respectively, we deduce

$$\begin{aligned}&\rho_{2n+2}\rho_{2n+3}\ R_n(x)\\&=\Big\{2\ \varepsilon\ D+8\ D\ x\ D\Big\}\cdot\Big\{(2+\varepsilon)\ \mathrm{I}_{\mathcal{P}}+2(4+\varepsilon)\ x\ D+8\ x\ D\ x\ D\Big\}\ \Big(R_{n+1}(x)\Big),\end{aligned}$$

with $n\in\mathbb{N}$, and also

$$\begin{aligned}&\rho_{2n+1}\rho_{2n+2}\ P_n(x)\\&=\Big\{(2-\varepsilon)\ \mathrm{I}_{\mathcal{P}}-2(4-\varepsilon)\ D\ x+8\ D\ x\ D\ x\Big\}\cdot\Big\{2\ \varepsilon\ D+8\ D\ x\ D\Big\}\ \Big(P_{n+1}(x)\Big),\end{aligned}$$

These last two identities may be easily transformed after simple calculations into (3.4.4)-(3.4.3), respectively, bearing in mind (3.4.7)-(3.4.8) and (3.4.10)-(3.4.9).

Likewise, by means of simple manipulations, the system of equalities (3.4.13) and (3.4.14) gives rise to another system of two equalities: one involving exclusively elements of the set of polynomials $\{b_n\}_{n\in\mathbb{N}}$ and the other having only elements of the set of polynomials $\{a_n\}_{n\in\mathbb{N}}$, which, on account of the identities (3.4.15), may be transformed into the following equalities

$$\begin{aligned}&\rho_{2n+2}\rho_{2n+3}\ b_n(x)\\&=\Big\{(2-\varepsilon)\ \mathrm{I}_{\mathcal{P}}-2(4-\varepsilon)\ D\ x+8\ D\ x\ D\ x\Big\}\cdot\Big\{2\ \varepsilon\ D+8\ D\ x\ D\Big\}\ \Big(b_{n+1}(x)\Big),\end{aligned}\tag{3.4.16}$$

for any $n\in\mathbb{N}$ and

$$\begin{aligned}&\rho_{2n+1}\rho_{2n+2}\ a_{n-1}(x)\\&=\Big\{2\ \varepsilon\ D+8\ D\ x\ D\Big\}\cdot\Big\{(2+\varepsilon)\ \mathrm{I}_{\mathcal{P}}+2(4+\varepsilon)\ x\ D+8\ x\ D\ x\ D\Big\}\ \Big(a_n(x)\Big),\end{aligned}\tag{3.4.17}$$

for any $n\in\mathbb{N}$ and with $a_{-1}=0$. The relation (3.4.16) gives rise to (3.4.6), whereas the relation (3.4.17) with $n$ replaced by $n+1$ leads to (3.4.5), under the definitions (3.4.7)-(3.4.10). □

More information concerning the two polynomial sets $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ is provided in the next result. The reader who may not be interested in these technicalities should go directly to the next section.

**Proposition 3.4.2.** *Let $\{B_n\}_{n\in\mathbb{N}}$ be a $\mathcal{F}_\varepsilon$Appell sequence, quadratically decomposed according to (3.0.1)-(3.0.2). Then either $\{B_n\}_{n\in\mathbb{N}}$ is symmetric or there exists an integer $p \geqslant 0$ such that $a_p(\cdot) \neq 0$ (respectively, $b_p(\cdot) \neq 0$). In this case, we have*

$$a_n(x) = 0,\ b_n(x) = 0, \quad \textit{with } 0 \leqslant n \leqslant p-1, \ \textit{when} \ \ p \geqslant 1, \tag{3.4.18}$$

$$a_{p+n}(x) = \binom{n+p+1}{n} \frac{\left(p+\frac{3}{2}\right)_n \left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n \left(p+2+\frac{\varepsilon}{4}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_n \left(1+\frac{\varepsilon}{4}\right)_n} a_p \ \widehat{a}_n(x), \tag{3.4.19}$$

$$for\ n \in \mathbb{N},$$

$$b_{p+n}(x) = \binom{n+p}{n} \frac{\left(p+\frac{3}{2}\right)_n \left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n \left(p+1+\frac{\varepsilon}{4}\right)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)_n \left(1+\frac{\varepsilon}{4}\right)_n} b_p \ \widehat{b}_n(x)\,, \tag{3.4.20}$$

$$for\ n \in \mathbb{N},$$

*where the two polynomials $\widehat{a}_n$ and $\widehat{b}_n$ are satisfy the condition $\deg \widehat{a}_n(x) = n$ and $\deg \widehat{b}_n(x) = n,\ n \in \mathbb{N},$; as usual, $(a)_n = a\,(a+1)\ldots(a+n-1)$ represents the Pochhammer symbol.*

*Proof.* If $\{B_n\}_{n\in\mathbb{N}}$ is a symmetric sequence then $a_n(\cdot) = 0,\ n \in \mathbb{N}$, and also $b_n(\cdot) = 0,\ n \in \mathbb{N}$. Reciprocally, if $a_n(\cdot) = 0,\ n \in \mathbb{N}$ (respectively, $b_n(\cdot) = 0,\ n \in \mathbb{N}$), then, from (3.4.13), $b_n(\cdot) = 0,\ n \in \mathbb{N}$ (respectively $a_n(\cdot) = 0,\ n \in \mathbb{N}$, from (3.4.14) ).
When $\{B_n\}_{n\in\mathbb{N}}$ is not a symmetric sequence, let $p \geqslant 0$ be the smallest integer such that $a_p(\cdot) \neq 0$ and $a_n(\cdot) = 0,\ 0 \leqslant n \leqslant p-1$ when $p \geqslant 1$ . From (3.4.14), we have $b_n(\cdot) =$ constant $= b_n,\ 0 \leqslant n \leqslant p$ and by virtue of (3.4.13), $b_n(\cdot) = 0$ for $0 \leqslant n \leqslant p-1$, $\rho_{2p+2}\, b_p(x) = (2+\varepsilon)\, a_p(x) + 2(8+\varepsilon)\, x\, a_p'(x) + 8\, x^2\, a_p''(x)$, which implies $a_p(\cdot) =$ constant $=$ $a_p \neq 0$. Thus, $(2+\varepsilon)\, a_p = \rho_{2p+2}\, b_p$.
Proceeding by finite induction, by taking into account (3.4.13)-(3.4.14), we achieve the conclusion $\deg(a_{n+p}) = n$ and $\deg(b_{n+p}) = n,\ n \in \mathbb{N}$. Therefore we may consider two nonzero sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} a_{n+p}(x) &= \lambda_n\, \widehat{a}_n(x)\ , \\ b_{n+p}(x) &= \mu_n\, \widehat{b}_n(x)\ , n \in \mathbb{N}, \end{aligned} \tag{3.4.21}$$

where $\widehat{a}_n(\cdot)$ and $\widehat{b}_n(\cdot)$ represent two monic polynomials of degree $n \in \mathbb{N}$, $\mu_0 = b_p$ and $\lambda_0 = a_p$ .

Replacing in (3.4.13) and (3.4.14) $n$ by $n+p$ and taking into account (3.4.21), we obtain

$$\rho_{2n+2p+2}\,\mu_n\,\widehat{b}_n(x) = (2+\varepsilon)\,\lambda_n\,\widehat{a}_n(x) + 2(8+\varepsilon)\,x\,\lambda_n\,\widehat{a}_n{}'(x) + 8\,x^2\,\lambda_n\,\widehat{a}_n{}''(x), n\in\mathbb{N},$$
$$\rho_{2n+2p+1}\,\lambda_{n-1}\,\widehat{a}_{n-1}(x) = 2(4+\varepsilon)\,\mu_n\,\widehat{b}_n{}'(x) + 8\,x\,\mu_n\,\widehat{b}_n{}''(x), \quad n\in\mathbb{N}.$$

Consequently the nonzero sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ satisfy the system

$$\rho_{2n+2p+2}\,\mu_n = 8\,\left(n+\frac{1}{2}\right)\,\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)\,\lambda_n\,, \quad n\in\mathbb{N},$$
$$\rho_{2n+2p+1}\,\lambda_{n-1} = 8\,n\,\left(n+\frac{\varepsilon}{4}\right)\,\mu_n, \quad n\in\mathbb{N}.$$

which implies

$$\rho_{2n+2p+2}\,\mu_n = 8\,\left(n+\frac{1}{2}\right)\,\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)\,\lambda_n\,, \quad n\in\mathbb{N},$$
$$\rho_{2n+2p+3}\,\rho_{2n+2p+4}\,\lambda_n = 64\,(n+1)\,\left(n+1+\frac{\varepsilon}{4}\right)\left(n+\frac{3}{2}\right)\,\left(n+\frac{3}{2}+\frac{\varepsilon}{4}\right)\lambda_{n+1}, n\in\mathbb{N},$$

and, because $\rho_{n+1} = (n+1)\,\big(2(n+1)+\varepsilon\big),\ n\in\mathbb{N}$, it yields

$$\mu_n = \frac{\left(n+\frac{1}{2}\right)\,\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)}{(n+p+1)\left(n+p+1+\frac{\varepsilon}{4}\right)}\,\lambda_n\,, \quad n\in\mathbb{N},$$
$$\lambda_{n+1} = \binom{n+p+2}{n+1}\frac{(p+\frac{3}{2})_{n+1}\,\left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_{n+1}\,\left(p+2+\frac{\varepsilon}{4}\right)_{n+1}}{\left(\frac{3}{2}\right)_{n+1}\,\left(1+\frac{\varepsilon}{4}\right)_{n+1}\,\left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_{n+1}}\,\lambda_0, \quad n\in\mathbb{N},$$

where $(y)_k$ represents the *Pochhammer* symbol. Finally we achieve,

$$\lambda_n = \binom{n+p+1}{n}\,\frac{\left(p+\frac{3}{2}\right)_n\,\left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\,\left(p+2+\frac{\varepsilon}{4}\right)_n}{\left(\frac{3}{2}\right)_n\,\left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\,\left(1+\frac{\varepsilon}{4}\right)_n}\,\lambda_0, \quad n\in\mathbb{N},$$
$$\mu_n = \frac{\left(n+\frac{1}{2}\right)\,\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)}{(n+p+1)\,\left(n+p+1+\frac{\varepsilon}{4}\right)}\,\lambda_n\,, \quad n\in\mathbb{N},$$

whence the result. □

The two MPS that came up with the quadratic decomposition of an $\mathcal{F}_\varepsilon$-Appell sequence, are indeed also Appell sequences with respect to the lowering operators $\mathcal{G}_{\varepsilon,1}$ and $\mathcal{G}_{\varepsilon,-1}$, regarding definition 3.0.6. Analogously, on account of the relations (3.4.5)-(3.4.6) and (3.4.19)-(3.4.20) given in proposition 3.4.2, we may say that the sequences $\{\widehat{a}_n\}_{n\in\mathbb{N}}$ and $\{\widehat{b}_n\}_{n\in\mathbb{N}}$ are, respectively, $\mathcal{G}_{\varepsilon,1}$ and $\mathcal{G}_{\varepsilon,-1}$-Appell. The study of these arising Appell sequences will proceed henceforth as a whole rather than individually, so, under the particular choices of $\mu=-1$ or $\mu=1$, they may be viewed as Appell sequences with respect to the lowering operator

$$\mathcal{G}_{\varepsilon,\mu} := \Big(4DxD+\varepsilon D\Big)\Big(8(xD)^2+2\varepsilon xD+2\mathrm{I}_{\mathcal{P}}+\mu\,\left(8xD+\varepsilon\mathrm{I}_{\mathcal{P}}\right)\Big)$$

with the convention: $(xD)^{k+1} = xD\,(xD)^k$ for any integer $k \geqslant 0$. Naturally, it is possible to express:

$$\begin{aligned} \mathcal{G}_{\varepsilon,\mu} \;:=\;& 32\,D(xD)^3 + 16\varepsilon\,D(xD)^2 + 2(4+\varepsilon^2)\,DxD + 2\varepsilon\,D \\ & +\mu\Big\{32\,D(xD)^2 + 12\varepsilon\,DxD + \varepsilon^2\,D\Big\}, \end{aligned} \tag{3.4.22}$$

The forthcoming developments will be made from a functional point of view, requiring the characterisation of the corresponding dual sequence, which will be carried out in the next section.

## 3.5 The $\mathcal{G}_{\varepsilon,\mu}$-Appell sequences

From a given a MPS $\{B_n\}_{n\in\mathbb{N}}$ it is possible to construct another MPS $\{B_n^{[1]}(\cdot;\mathcal{G}_{\varepsilon,\mu})\}_{n\in\mathbb{N}}$ through

$$B_n^{[1]}(x\,;\mathcal{G}_{\varepsilon,\mu}) = \frac{1}{\widehat{\rho}_{n+1}}\Big(\mathcal{G}_{\varepsilon,\mu}\,B_{n+1}\Big)(x)\,,\; n\in\mathbb{N} \tag{3.5.1}$$

where $\mathcal{G}_{\varepsilon,\mu}$ is given by (3.4.22) and

$$\begin{aligned} &\widehat{\rho}_{n+1} := \widehat{\rho}_{n+1}(\varepsilon,\mu) \\ &= (n+1)\,\big(4(n+1)+\varepsilon\big)\,\Big(2 + 2(n+1)\big(4(n+1)+\varepsilon\big) + (8+8n+\varepsilon)\mu\Big) \end{aligned} \tag{3.5.2}$$

for $n\in\mathbb{N}$. Necessarily, the parameters $\varepsilon$ and $\mu$ must be chosen so that $\widehat{\rho}_{n+1}\neq 0$, for all the integers $n\geqslant 0$, therefore, $\varepsilon$ and $\mu$ are two complex parameters such that

$$\varepsilon \neq -4(n+1) \qquad \text{and} \qquad \mu \neq -\frac{2+2(n+1)(4n+4+\varepsilon)}{8(n+1)+\varepsilon}\,, \qquad n\in\mathbb{N}. \tag{3.5.3}$$

Whenever $\mu\in\{-1,1\}$, then $\widehat{\rho}_{n+1}(\varepsilon,\mu)$ equals $\eta_{n+1}(\varepsilon,\mu)$, given by (3.4.9)-(3.4.10), for any integer $n\in\mathbb{N}$.

The characterisation of the $\mathcal{G}_{\varepsilon,\mu}$-Appell sequences, will be taken my means of the properties of the corresponding dual sequence. For this purpose we shall previously know more about the transpose ${}^t\mathcal{G}_{\varepsilon,\mu}$ of $\mathcal{G}_{\varepsilon,\mu}$. On the basis of (1.1.2) and(1.1.4), we have:

$$\begin{aligned}
\langle {}^t\mathcal{G}_{\varepsilon,\mu} u\ ,\ f\rangle &= \langle u\ ,\ \mathcal{G}_{\varepsilon,\mu} f\rangle \\
&= \Big\langle u\ ,\ \big\{32\, D(xD)^3 + 16(\varepsilon+2\mu)\, D(xD)^2 \\
&\quad +2(4+\varepsilon^2+6\,\varepsilon\,\mu)\, DxD + \varepsilon(2+\varepsilon\,\mu)\, D\big\} f \Big\rangle \\
&= \Big\langle \big\{32\, D(xD)^3 - 16(\varepsilon+2\mu)\, D(xD)^2 \\
&\quad +2(4+\varepsilon^2+6\,\varepsilon\,\mu)\, DxD - \varepsilon(2+\varepsilon\,\mu)\, D\big\} u, f \Big\rangle \\
&= \big\langle \mathcal{G}_{-\varepsilon,-\mu}\, u, f \big\rangle
\end{aligned}$$

therefore

$$\begin{aligned}
{}^t\mathcal{G}_{\varepsilon,\mu} &= 32\, D(xD)^3 - 16(\varepsilon+2\mu)\, D(xD)^2 \\
&\quad +2(4+\varepsilon^2+6\,\varepsilon\,\mu)\, DxD - \varepsilon\,(2+\varepsilon\,\mu)\, D\,.
\end{aligned}$$

Hence, convention on $D$ ( ${}^tD = -D$) permits to write ${}^t\alpha_\nu := (-1)^{\nu+1} D(xD)^\nu$ , with $\alpha_\nu := D(xD)^\nu$, leaving out a slight abuse of notation without consequence. Thus ${}^t\mathcal{G}_{\varepsilon,\mu} := \mathcal{G}_{-\varepsilon,-\mu}$ and $\mathcal{G}_{\varepsilon,\mu}$ is defined on $\mathcal{P}$ and $\mathcal{P}'$.

For the sequel, it is worthy to express $\mathcal{G}_{\varepsilon,\mu}$ in terms of $x^k\, D^{k+1}$ instead of $D(xD)^k$ (with $k=0,1,2,3$). The identities

$$\begin{aligned}
DxD &= x\,D^2 + D \\
D(xD)^2 &= x^2\,D^3 + 3\,x\,D^2 + D \\
D(xD)^3 &= x^3\,D^4 + 6\,x^2\,D^3 + 7\,x\,D^2 + D\,.
\end{aligned}$$

permit to express the operator $\mathcal{G}_{\varepsilon,\mu}$ given by (3.4.22) as follows:

$$\begin{aligned}
\mathcal{G}_{\varepsilon,\mu} &= 32\,x^3\,D^4 + 16(12+\varepsilon)\,x^2\,D^3 \\
&\quad +2\big(116+\varepsilon(24+\varepsilon)\big)\,x\,D^2 + 2(4+\varepsilon)(5+\varepsilon)\,D \\
&\quad +\mu\Big\{32\,x^2\,D^3 + 12(8+\varepsilon)\,x\,D^2 + (4+\varepsilon)(8+\varepsilon)\,D\Big\}\,.
\end{aligned} \tag{3.5.4}$$

and, by means of simple computations, we are able to deduce the $\mathcal{G}_{\varepsilon,\mu}$-derivative of the product of two polynomials:

$$\begin{aligned}
\mathcal{G}_{\varepsilon,\mu}\big(f\,p\big)(x) &= f(x)\,\big(\mathcal{G}_{\varepsilon,\mu}\,p\big)(x) + \big(\mathcal{G}_{\varepsilon,\mu}\,f\big)(x)\,p(x) \\
&\quad +128\,x^3\,f'(x)\,p^{(3)}(x) + 48\Big\{(\varepsilon+12+2\mu)f'(x) \\
&\quad +4xf''(x)\Big\}x^2\,p''(x) + \Big\{\big(116+\varepsilon^2+48\mu+6\varepsilon(4+\mu)\big)f'(x) \\
&\quad +12(\varepsilon+2(6+\mu))x\,f''(x) + 32\,x^2 f^{(3)}(x)\Big\}4\,x\,p'(x)
\end{aligned} \tag{3.5.5}$$

for any $p, f \in \mathcal{P}$. By transposition, we may also compute the $\mathcal{G}_{\varepsilon,\mu}$-derivative of the product of a polynomial by a form:

$$\begin{aligned}\mathcal{G}_{-\varepsilon,-\mu}\big(fu\big) = \; & f\,\mathcal{G}_{-\varepsilon,-\mu}\big(u\big) - \mathcal{G}_{\varepsilon,\mu}\big(f\big)\; u + f'(x)\; L_3(u) + f''(x)\; L_2(u)\\ & + f^{(3)}(x)\; L_1(u) + 2^6\; x^3\; f^{(4)}(x) u \quad , \qquad f \in \mathcal{P}\,,\; u \in \mathcal{P}'\,,\end{aligned} \tag{3.5.6}$$

where

$$\begin{aligned} L_3(u) &= \tau_{3,0}\; u + \tau_{3,1}\; x\; u' + \tau_{3,2}\; x^2\; u'' + 2^7 \cdot x^3\; (u)^{(3)}\\ L_2(u) &= \tau_{2,0}\; x\; u + \tau_{2,1}\; x^2\; u' + 3 \cdot 2^6\; x^3\; u'' \\ L_1(u) &= \tau_{1,0}\; x^2\; u + 2^7\; x^3\; u' \end{aligned} \tag{3.5.7}$$

with

$$\begin{aligned} \tau_{3,0} &= 4(20 + \varepsilon^2 + 6\varepsilon\mu) & \tau_{2,0} &= 2^2\,\big(116 + \varepsilon^2 + 6\varepsilon\mu\big)\\ \tau_{3,1} &= 2^2\,\big(116 + \varepsilon^2 + 6\varepsilon(\mu - 4) - 48\mu\big) & \tau_{2,1} &= 2^4 \cdot 3\,\big(12 - \varepsilon - 2\mu\big)\\ \tau_{3,2} &= -2^4 \cdot 3\,\big(\varepsilon - 12 + 2\mu\big) & \tau_{1,0} &= 2^7 \cdot 3 \end{aligned}$$

As usual, we will denote by $\{u_n\}_{n\in\mathbb{N}}$ the dual sequence of $\{B_n\}_{n\in\mathbb{N}}$. To maintain the coherence, the dual sequence associated to the MPS $\{B_n^{[1]}(\cdot;\mathcal{G}_{\varepsilon,\mu})\}_{n\in\mathbb{N}}$ will be denoted by $\{u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\}_{n\in\mathbb{N}}$.

**Lemma 3.5.1.** *The dual sequence of $\{B_n^{[1]}(\cdot;\mathcal{G}_{\varepsilon,\mu})\}_{n\in\mathbb{N}}$ denoted as $\{u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\}_{n\in\mathbb{N}}$ and the dual sequence $\{u_n\}_{n\in\mathbb{N}}$ associated to $\{B_n\}_{n\in\mathbb{N}}$ are related through*

$$\mathcal{G}_{-\varepsilon,-\mu}\big(u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\big) = \widehat{\rho}_{n+1}\; u_{n+1}, \quad n \in \mathbb{N}, \tag{3.5.8}$$

*where $\widehat{\rho}_{n+1}$, $n \in \mathbb{N}$, is given by (3.5.2).*

*Proof.* Indeed, successively we have

$$\begin{aligned} &\langle u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu}), B_m^{[1]}(x;\mathcal{G}_{\varepsilon,\mu})\rangle = \delta_{n,m}\,, \quad n,m \geqslant 0,\\ &\langle u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu}), \mathcal{G}_{\varepsilon,\mu}(B_{m+1})\rangle = \widehat{\rho}_{n+1}(\varepsilon,\mu)\,\delta_{n,m}\,, \quad n,m \geqslant 0,\\ &\langle \mathcal{G}_{-\varepsilon,-\mu}\,\big(u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\big), B_{m+1}\rangle = \widehat{\rho}_{n+1}(\varepsilon,\mu)\,\delta_{n,m}\,, \quad n,m \geqslant 0. \end{aligned} \tag{3.5.9}$$

In particular,

$$\langle \mathcal{G}_{-\varepsilon,-\mu}\big(u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\big), B_{m+1}\rangle = 0, \quad m \geqslant n+1,\; n \in \mathbb{N},$$

which, due to lemma 1.3.1, implies

$$\mathcal{G}_{-\varepsilon,-\mu}\big(u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\big) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu}\; u_\nu, \quad n \in \mathbb{N},$$

with $\lambda_{n,\nu} = \langle \mathcal{G}_{-\varepsilon,-\mu}\big(u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\big), B_\nu\rangle,\ 0 \leqslant \nu \leqslant n+1$. Consequently, because of (3.5.9), we obtain (3.5.8). □

This last result enables us to express all the elements of the dual sequence in terms of the first one:

**Proposition 3.5.2.** *The MPS $\{B_n\}_{n\in\mathbb{N}}$ is a $\mathcal{G}_{\varepsilon,\mu}$-Appell sequence if and only if its dual sequence $\{u_n\}_{n\in\mathbb{N}}$ fulfils*

$$u_n = \frac{1}{\alpha_n}\, \mathcal{G}^{\,n}_{-\varepsilon,-\mu}(u_0), \quad n \in \mathbb{N}, \tag{3.5.10}$$

*where*

$$\alpha_n = 32^n\ n!\ \Big(1+\frac{\varepsilon}{4}\Big)_n \left(\frac{8+\varepsilon+4\mu-\Delta_{\varepsilon,\mu}}{8}\right)_n \left(\frac{8+\varepsilon+4\mu+\Delta_{\varepsilon,\mu}}{8}\right)_n,\ n \in \mathbb{N},$$

*with $\Delta_{\varepsilon,\mu} = \sqrt{\varepsilon^2+16(\mu^2-1)}$, and $\mathcal{G}^{\,n}_{-\varepsilon,-\mu}$ represents the $n^{th}$-power of the operator $\mathcal{G}_{-\varepsilon,-\mu}$.*

*Proof.* The condition is necessary. From (3.5.8), the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies

$$\mathcal{G}_{-\varepsilon,-\mu}(u_n) = \widehat{\rho}_{n+1}(\varepsilon,\mu)\ u_{n+1}, \quad n \in \mathbb{N}, \tag{3.5.11}$$

with $\widehat{\rho}_{n+1}(\varepsilon,\mu)$ as given in (3.5.2). In particular, for $n=0$,

$$u_1 = \frac{1}{(4+\varepsilon)(10+8\mu+\varepsilon\ (2+\mu))}\ \mathcal{G}_{-\varepsilon,-\mu}\, u_0\ .$$

By recurrence, we get (3.5.10).

The condition is sufficient. From (3.5.10), it is easy to see that (3.5.11) is fulfilled. Therefore by comparing it with (3.5.8), we obtain

$$\mathcal{G}_{-\varepsilon,-\mu}\ \big(u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu})\big) = \mathcal{G}_{-\varepsilon,-\mu}\, u_n,\ n \in \mathbb{N}\,.$$

The lowering operator $\mathcal{G}_{-\varepsilon,-\mu}$ satisfies $\mathcal{G}_{-\varepsilon,-\mu}(\mathcal{P}) = \mathcal{P}$, and therefore $\mathcal{G}_{-\varepsilon,-\mu}$ is one-to-one on $\mathcal{P}'$. We then get $u_n^{[1]}(\mathcal{G}_{\varepsilon,\mu}) = u_n\ ,\ n \in \mathbb{N}$, whence, the expected result. □

## 3.6 About the orthogonality of a $\mathcal{G}_{\varepsilon,\mu}$-Appell sequence

This section aims to find all the polynomial sequences, if they exist, that are both orthogonal and $\mathcal{G}_{\varepsilon,\mu}$-Appell. A somewhat unexpected result occurs:

**Theorem 3.6.1.** *There is no regularly orthogonal polynomial sequence being $\mathcal{G}_{\varepsilon,\mu}$-Appell.*

In the absence of a better procedure, we follow the steps of the proof of theorem 3.6.1, which unfortunately makes this proof more technical than we wished.

*Proof.* Suppose there is a MOPS $\{B_n\}_{n\in\mathbb{N}}$ being also a $\mathcal{G}_{\varepsilon,\mu}$-Appell sequence. Therefore it fulfils the second order recurrence relation

$$\begin{aligned} &B_0(x)=1 \quad ; \quad B_1(x)=x-\beta_0 \\ &B_{n+2}(x)=(x-\beta_{n+1})B_{n+1}(x)-\gamma_{n+1}B_n(x)\,, \quad n\in\mathbb{N}, \end{aligned}$$

and the corresponding dual sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies $u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0,\ n \in \mathbb{N}$. Combining this last equality with (3.5.11), we deduce

$$\mathcal{G}_{-\varepsilon,-\mu}(B_n u_0)=\lambda_n\, B_{n+1}u_0, \quad n\in\mathbb{N}, \tag{3.6.1}$$

with

$$\lambda_n := \lambda_n(\varepsilon)=\frac{\widehat{\rho}_{n+1}(\varepsilon,\mu)}{\gamma_{n+1}}, \quad n\in\mathbb{N}, \tag{3.6.2}$$

where $\widehat{\rho}_{n+1},\ n\in\mathbb{N}$, is defined in (3.5.2). We recall that (3.5.2) is always different from zero because the parameters $\varepsilon$ and $\mu$ obey the conditions (3.5.3). The particular choice of $n=0$ in (3.6.1), provides

$$\mathcal{G}_{-\varepsilon,-\mu}\, u_0=\lambda_0\, B_1\, u_0\ . \tag{3.6.3}$$

Consider $n+1$ instead of $n$ in (3.6.1). Following (3.5.6)-(3.5.7), then, because of the $\mathcal{G}_{\varepsilon,\mu}$-Appell character and on account of (3.6.3), we derive

$$\begin{aligned} &B'_{n+1}\, L_3(u_0)+B''_{n+1}\, L_2(u_0)+B^{(3)}_{n+1}\, L_1(u_0) \\ &=\left\{\lambda_{n+1}\, B_{n+2}-\lambda_0\, B_1\, B_{n+1}+\lambda_n\, \gamma_{n+1}\, B_n-2^6\, x^3\, B^{(4)}_{n+1}\right\} u_0\ , \qquad n\in\mathbb{N}, \end{aligned} \tag{3.6.4}$$

In particular, considering $n=0$ in this last relation, $u_0$ fulfils the equality:

$$L_3(u_0)=U_2(x)\, u_0 \tag{3.6.5}$$

where $L_3(u_0)$ is given in (3.5.7) and

$$U_2(x)=\lambda_1 B_2(x)-\lambda_0 B_1^{\,2}(x)+\lambda_0\, \gamma_1\ .$$

On account of (3.6.5), the relation (3.6.4) becomes

$$\begin{aligned} &B''_{n+1}\, L_2(u_0)+B^{(3)}_{n+1}\, L_1(u_0) \\ &=\left\{\lambda_{n+1}\, B_{n+2}-\lambda_0\, B_1\, B_{n+1}+\lambda_n\, \gamma_{n+1}\, B_n-U_2\, B'_{n+1}-2^6\, x^3\, B^{(4)}_{n+1}\right\} u_0,\ n\in\mathbb{N}, \end{aligned} \tag{3.6.6}$$

and when $n = 1$, this relation becomes

$$L_2(u_0) = U_3(x)\, u_0 \tag{3.6.7}$$

where $L_2(u_0)$ is given by (3.5.7) and

$$U_3(x) = \frac{1}{2}\left\{\lambda_2 B_3(x) - \lambda_0 B_1(x)\, B_2(x) + \lambda_1\, \gamma_2\, B_1(x) - B_2'(x)\, U_2(x)\right\}.$$

Therefore, due to (3.6.7), the relation (3.6.6) may be transformed into

$$\begin{aligned} B^{(3)}_{n+1}\, L_1(u_0) = \Big\{ & \lambda_{n+1}\, B_{n+2} - \lambda_0\, B_1\, B_{n+1} + \lambda_n\, \gamma_{n+1}\, B_n \\ & -B'_{n+1}\, U_2 - B''_{n+1}\, U_3 - 2^6\, x^3\, B^{(4)}_{n+1} \Big\}\, u_0\,, \qquad n \in \mathbb{N}, \end{aligned} \tag{3.6.8}$$

and taking $n = 2$ we obtain:

$$L_1(u_0) = U_4(x)\, u_0 \tag{3.6.9}$$

where $L_1(u_0)$ is given in (3.5.7) and

$$U_4(x) = \frac{1}{6}\left\{\lambda_3 B_4(x) - \lambda_0 B_1(x)\, B_3(x) + \lambda_2\, \gamma_3\, B_2(x) - B_3'(x)\, U_2(x) - B_3''(x)\, U_3(x)\right\}.$$

Naturally, $\deg U_k \leqslant k$ for $k = 2, 3$ or $4$, so, there are coefficients $\theta_{k,j}$ with $0 \leqslant j \leqslant k$ such that

$$U_k(x) = \sum_{j=0}^{k} \theta_{k,j}\, x^j, \quad k = 2, 3, 4. \tag{3.6.10}$$

A single differentiation on both sides of (3.6.9) leads to

$$2^7\, x^3\, u_0'' + \left\{(3 \cdot 2^7 + \tau_{1,0})\, x^2 - U_4(x)\right\} u_0' = \left\{U_4'(x) - 2\, \tau_{1,0}\, x\right\} u_0\,. \tag{3.6.11}$$

Between (3.6.11) and (3.6.7) it is possible to eliminate the term in $u_0''$, and consequently we have

$$\begin{aligned} & \left\{(3^2 \cdot 2^8 + 3\, \tau_{1,0} - 2\, \tau_{2,1})\, x^2 - 3\, U_4(x)\right\} u_0' \\ & = \left\{3\, U_4'(x) - 2\, U_3(x) - 2\, (3\, \tau_{1,0} - \tau_{2,0})\, x\right\} u_0 \end{aligned} \tag{3.6.12}$$

The elimination of the term $u_0'$ between the equalities (3.6.12) and (3.6.9) leads to $C_3(x)\, u_0 = 0$ where

$$\begin{aligned} C_3(x) = & -2^7\, x^3\, \left\{3\, U_4'(x) - 2\, U_3(x) - 2\, (3\, \tau_{1,0} - \tau_{2,0})\, x\right\} \\ & + \left\{(3^2 \cdot 2^7 + 3\, \tau_{1,0} - 2\, \tau_{2,1})\, x^2 - 3\, U_4(x)\right\} \left\{U_4(x) - \tau_{1,0}\, x^2\right\} \end{aligned}$$

The regularity of $u_0$ permits to conclude $C_3(\cdot) = 0$, that is, $C_3$ has all its coefficients in $x$ identically zero. Taking into account the definition of the polynomials $U_k$ with $k = 3, 4$ presented in (3.6.10), we realise that $\deg C_3 \leqslant 8$ and we also achieve:

$$\theta_{4,4} = \theta_{4,0} = \theta_{4,1} = 0 \tag{3.6.13}$$

As a consequence, $C_3(x) = \sum_{j=3}^{6} c_{3,j}\; x^j$ and the conditions $c_{3,j} = 0$ for $j = 3, 4, 5, 6$ provide

$$\begin{aligned}
&\theta_{3,0} = 0 \quad , \quad \theta_{3,3} = \frac{3}{2^8}\; \theta_{4,3}^2 \quad , \quad \theta_{3,2} = \frac{1}{2^7}\, \theta_{4,3}\, (3\, \theta_{4,2} - 3\, \tau_{1,0} + \tau_{2,1}) \\
&\theta_{3,1} \;=\; \frac{1}{2^8}\, \big\{ 3\, \theta_{4,2}^2 + 2^8\, \tau_{2,0} - \theta_{4,2} \big(2^7 \cdot 3 + 6\, \tau_{1,0} - 2\, \tau_{2,1}\big) \\
&\qquad\quad -\tau_{1,0}(-2^7 \cdot 3 - 3\, \tau_{1,0} + 2\, \tau_{2,1}) \big\},
\end{aligned} \tag{3.6.14}$$

whence, $U_4(x) = \big(\theta_{4,3}\; x + \theta_{4,2}\big)\; x^2$ et $U_3(x) = \theta_{3,3}\; x^3 + \theta_{3,2}\; x^2 + \theta_{3,1}\; x$ .

Besides, differentiating both sides of (3.6.7) and then eliminating the term in $u_0^{(3)}$ between the resulting equation and (3.6.5), we deduce

$$\begin{aligned}
&\big\{ 2^7 \cdot 3^2 + 2\, \tau_{2,1} - 3\, \tau_{3,2} \big\}\; x^2\; u_0'' + \big\{ (2\, \tau_{2,0} + 4\, \tau_{2,1} - 3\, \tau_{3,1})\; x - 2\, U_3(x) \big\}\; u_0' \\
&= \big\{ -2\, \tau_{2,0} + 3\, \tau_{3,0} + 2\, U_3'(x) - 3\, U_2(x) \big\}\; u_0
\end{aligned} \tag{3.6.15}$$

Proceeding to the elimination of the term in $u_0''$ between (3.6.15) and (3.6.7), we get:

$$\begin{aligned}
&\big\{ \big[ 2^7 \cdot 3\, \tau_{2,0} - 2^6 \cdot 3^2\, \tau_{3,1} + \tau_{2,1}(-2^7 \cdot 3 - 2\, \tau_{2,1} + 3\, \tau_{3,2}\,)\, \big] x - 2^7 \cdot 3\, U_3(x) \big\}\; x\; u_0' \\
&= \big\{ \tau_{2,0}\, \big( \,2\, (2^7 \cdot 3 + \tau_{2,1}) - 3\, \tau_{3,2} \big)\; x - \big( 2^7 \cdot 3^2 + 2\, \tau_{2,1} - 3\, \tau_{3,2} \big)\; U_3(x) \\
&+3 \cdot 2^6\, \big( \,3\, \tau_{3,0} - 3\, U_2(x) + 2\, U_3'(x) \big)\; x \big\}\; u_0
\end{aligned} \tag{3.6.16}$$

By eliminating the term in $u_0'$ between (3.6.16) and (3.6.9), and by taking into consideration the regularity of $u_0$, we get the condition: $C_2 \equiv 0$ where

$$\begin{aligned}
C_2(x) = \;\; & -(2^7\; x^3) \Big\{ \tau_{2,0}\, \big( \,2\, (2^7 \cdot 3 + \tau_{2,1}) - 3\, \tau_{3,2} \big)\; x - \big( 2^7 \cdot 3^2 + 2\, \tau_{2,1} - 3\, \tau_{3,2} \big)\; U_3(x) \\
& +3 \cdot 2^6\, \big( \,3\, \tau_{3,0} - 3\, U_2(x) + 2\, U_3'(x) \big)\; x \Big\} \\
& + \Big\{ \big[ 2^7 \cdot 3\, \tau_{2,0} - 2^6 \cdot 3^2\, \tau_{3,1} + \tau_{2,1}(-2^7 \cdot 3 - 2\, \tau_{2,1} + 3\, \tau_{3,2}\,)\, \big] x^2 \\
& -2^7 \cdot 3\, U_3(x)\; x \Big\} \;\cdot\; \big\{ U_4(x) - \tau_{1,0}\; x^2 \big\}
\end{aligned} \tag{3.6.17}$$

After (3.6.13), we easily realise that the polynomial $C_2$ may be expressed as $C_2(x) = \sum_{j=4}^{7} c_{2,j}\, x^j$ . Due to (3.6.13)-(3.6.14), the condition $c_{2,7} = 0$ implies $\theta_{4,3} = 0$. According to (3.6.14), this yields

$$\begin{aligned} \theta_{3,0} = & \;\; 0 = \theta_{3,3} = \theta_{3,2} \\ \theta_{3,1} = & \;\; \frac{1}{2^8} \left\{ 3\, \theta_{4,2}^2 + 2^8\, \tau_{2,0} - \theta_{4,2} \big( 2^7 \cdot 3 + 6\, \tau_{1,0} - 2\, \tau_{2,1} \big) \right. \\ & \left. - \tau_{1,0} (-2^7 \cdot 3 - 3\, \tau_{1,0} + 2\, \tau_{2,1}) \right\} \end{aligned} \tag{3.6.18}$$

and, consequently, $U_3(x) = \theta_{3,1}\, x$ and $U_4(x) = \theta_{4,2}\, x^2$. From the conditions $c_{2,6} = 0 = c_{2,5}$ we deduce $\theta_{2,2} = \theta_{2,1} = 0$.

As a result, $U_2(x) = \theta_{2,0}$ , $U_3(x) = \theta_{3,1}\, x$ and $U_4(x) = \theta_{4,2}\, x^2$ , and, according to (3.6.9) $u_0$ fulfils

$$(\tau_{1,0} - \theta_{4,2})\, x^2\, u_0 + 2^7\, x^3\, u_0' = 0.$$

contradicting the regularity of $u_0$. □

Motivated by the impossibility of the existence of $\mathcal{G}_{\varepsilon,\mu}$-Appell sequences being also orthogonal, a research on the existence and subsequent determination of $d$-orthogonal $\mathcal{G}_{\varepsilon,\mu}$-Appell sequences appears to be an interesting problem, although also tricky to solve. We will not follow this path, in order to maintain some coherence in the concepts under research. Instead, based on theorem 3.3.1 and theorem 3.4.1, we will proceed to the complete description of the QD of a Laguerre classical sequence with parameter $\frac{\varepsilon}{2}$.

## 3.7 Applications. The quadratic decomposition of a Laguerre sequence

The quadratic decomposition of a non-symmetric sequence is far from being obvious, nevertheless, after the work of Maroni [80, 81] we have theoretical resources enabling to deal with this problem in a more straightforward manner.

However, based on some already known results as well as the obtained ones we are able to describe the associated polynomial sequences to the QD of a Laguerre sequence with complex parameter.

**Proposition 3.7.1.** *A Laguerre sequence $\{B_n\}_{n\in\mathbb{N}}$ of parameter $\frac{\varepsilon}{2}$ (with $\varepsilon \neq -2(n+1), n \in \mathbb{N}$) fulfils (3.0.1)-(3.0.2) where $\{R_n\}_{n\in\mathbb{N}}$ and $\{P_n\}_{n\in\mathbb{N}}$ are respectively $\mathcal{G}_{\varepsilon,1}$ and $\mathcal{G}_{\varepsilon,-1}$-Appell* *sequences and* $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ *are two PS given by*

$$a_n(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu} R_\nu(x), \quad n \in \mathbb{N} \tag{3.7.1}$$

$$b_n(x) = \sum_{\nu=0}^{n} \theta_{n,\nu} P_\nu(x), \quad n \in \mathbb{N}\,, \tag{3.7.2}$$

*with*

$$\lambda_{n,\nu} = \binom{2n+2}{2\nu} \frac{(-1)^{n-\nu}\, 2^{2n-2\nu+1}}{2\nu+1}\, \frac{\left(2+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(2+\frac{\varepsilon}{2}\right)_{2\nu}}\, \mathfrak{G}_{2n-2\nu+2}\,, \quad 0 \leqslant \nu \leqslant n, \quad n \in \mathbb{N}, \tag{3.7.3}$$

$$\theta_{n,\nu} = \binom{2n+2}{2\nu} \frac{(-1)^{n-\nu}\, 2^{2n-2\nu}}{n+1}\, \frac{\left(1+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(1+\frac{\varepsilon}{2}\right)_{2\nu}}\, \mathfrak{G}_{2n-2\nu+2}\,, \quad 0 \leqslant \nu \leqslant n, \quad n \in \mathbb{N}, \tag{3.7.4}$$

*where the symbol* $(a)_k = a(a+1)\ldots(a+k-1),\ k \geqslant 0,$ *denotes the* Pochhammer symbol *and* $\mathfrak{G}_n$ *represent the* unsigned Genocchi numbers.

*Genocchi numbers* were presumably introduced by Lucas [76], but they owe their name to the italian mathematician Angelo Genocchi (1817-1889) [46]. However, in a letter to Christian Goldbach (long before Genocchi or Lucas were born), Leonard Euler showed that he had already perceived the existence of such numbers. These numbers are intimately related to the much more famous *Bernoulli numbers* as it will be exposed just after the proof of the precedent result. Intensive studies on *Genocchi numbers* were developed by E.T. Bell in the 1920s in [10] and [11] and there are a lot of possibilities for computing their values (see for example Domaratzki [36], Ehrenborg and Steingrímsson [39] and Terrill and Terrill [103], and also the entry A036969 in OEIS Sloane [100] for further references).

In order to proceed with the development of the proof we need a description already known about the QD of a MOPS.

**Lemma 3.7.2.** *[80] Given a MPS* $\{B_n\}_{n\in\mathbb{N}}$*, it is possible to associate two MPS* $\{R_n\}_{n\in\mathbb{N}}$ *and* $\{P_n\}_{n\in\mathbb{N}}$ *and two sequences* $\{a_n\}_{n\in\mathbb{N}}$ *and* $\{b_n\}_{n\in\mathbb{N}}$ *according to (3.0.1)-(3.0.2) and (3.7.1)-(3.7.2). If, in addition,* $\{B_n\}_{n\in\mathbb{N}}$ *is a MOPS fulfilling the second order recurrence relation*

(1.4.3), *necessarily the coefficients* $\lambda_{n,\nu}, \theta_{n,\nu}$, $0 \leqslant \nu \leqslant n, n \in \mathbb{N}$, *satisfy the following system:*

$$\lambda_{n,n} = -\sum_{\nu=1}^{n} \{\beta_{2\nu} + \beta_{2\nu+1}\}, \quad n \in \mathbb{N}, \tag{3.7.5}$$

$$\theta_{n,n} = -\beta_0 - \sum_{\nu=1}^{n} \{\beta_{2\nu-1} + \beta_{2\nu}\}, \quad n \in \mathbb{N}, \tag{3.7.6}$$

$$\theta_{n+1,\nu} + \gamma_{2n+2}\theta_{n,\nu} = \lambda_{n,\nu-1} + \gamma_{2\nu+1}\lambda_{n,\nu} + \sum_{\mu=\nu}^{n} \lambda_{n,\mu}\theta_{\mu,\nu}\, \beta_{2\mu+1} \tag{3.7.7}$$

$$\lambda_{n+1,\nu} + \gamma_{2n+3}\lambda_{n,\nu} = \theta_{n+1,\nu} + \gamma_{2\nu+2}\theta_{n+1,\nu+1} + \sum_{\mu=\nu}^{n} \theta_{n+1,\mu+1}\, \lambda_{\mu,\nu}\, \beta_{2\mu+2} \tag{3.7.8}$$

*for* $0 \leqslant \nu \leqslant n$, $n \in \mathbb{N}$, *with* $\lambda_{n,-1} = 0, n \in \mathbb{N}$.

**Proof of proposition 3.7.1.** Let $\{B_n\}_{n\in\mathbb{N}}$ be a Laguerre sequence of parameter $\frac{\varepsilon}{2}$ with $\varepsilon \neq -2n$, $n \geqslant 1$. The author and Maroni have shown in [72, theorem 6] that such sequence is the unique MOPS being $\mathcal{F}_\varepsilon$-Appell. So, necessarily, the elements of $\{B_n\}_{n\in\mathbb{N}}$ satisfy the second order recurrence relation

$$\begin{array}{l} B_0(x) = 1 \quad ; \quad B_1(x) = x - \beta_0 \\ B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x)\ , \quad n \in \mathbb{N}, \end{array}$$

and, recalling the information given in Table 2.1, the corresponding recurrence coefficients $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$ are

$$\beta_n = 2n + 1 + \frac{\varepsilon}{2} \qquad ; \qquad \gamma_{n+1} = (n+1)\left(n + 1 + \frac{\varepsilon}{2}\right), \qquad n \in \mathbb{N}. \tag{3.7.9}$$

On the attempt of obtaining supplementary information about the polynomial sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ associated to the QD of $\{B_n\}_{n\in\mathbb{N}}$ as in (3.0.1)-(3.0.2), we consider the expansion of the elements of $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in terms of those of $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$, respectively, in accordance to (3.7.1)-(3.7.2). From this on, we are focused on obtaining explicit expressions for the elements of the two sets of numbers $\{\lambda_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ and $\{\theta_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ fulfilling the conditions (3.7.1)-(3.7.2).

By virtue of theorem 3.4.1, the MPS $\{R_n\}_{n\in\mathbb{N}}$ and $\{P_n\}_{n\in\mathbb{N}}$ are respectivelly $\mathcal{G}_{\varepsilon,1}$ and $\mathcal{G}_{\varepsilon,-1}$-Appell sequences. Just as it was observed in the proof of theorem 3.4.1, the conditions (3.4.11)-(3.4.14) hold. In particular from (3.4.13) and on account of (3.7.1)-(3.7.2), we derive

$$2\gamma_{2n+2}\sum_{\nu=0}^{n} \theta_{n,\nu}P_\nu(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu}\big\{(2+\varepsilon)\mathrm{I} + 2(8+\varepsilon)xD + 8x^2D^2\big\}R_\nu(x),\ n \in \mathbb{N}.$$

Due to (3.4.12), we have

$$\gamma_{2n+2}\sum_{\nu=0}^{n}\theta_{n,\nu}P_\nu(x)=\sum_{\nu=0}^{n}\lambda_{n,\nu}\gamma_{2\nu+1}P_\nu(x),\ n\in\mathbb{N},$$

which, because $\{P_n\}_{n\in\mathbb{N}}$ is an independent sequence, provides

$$\theta_{n,\nu}=\frac{\gamma_{2\nu+1}}{\gamma_{2n+2}}\,\lambda_{n,\nu}\,,\qquad n\in\mathbb{N},\quad 0\leqslant\nu\leqslant n. \tag{3.7.10}$$

On the other hand, (3.7.1)-(3.7.2) permits to write the relation (3.4.13) as follows:

$$2\gamma_{2n+1}\sum_{\nu=0}^{n}\lambda_{n-1,\nu}\,R_\nu(x)=\sum_{\nu=0}^{n-1}\theta_{n,\nu+1}\big\{2(4+\varepsilon)D+8xD^2\big\}P_{\nu+1}(x),\ n\geqslant 1.$$

The relation (3.4.11) allows to transform the previous into

$$\gamma_{2n+1}\sum_{\nu=0}^{n}\lambda_{n-1,\nu}\,R_\nu(x)=\sum_{\nu=0}^{n-1}\theta_{n,\nu+1}\,\gamma_{2\nu+2}\,R_\nu(x),\ n\geqslant 1,$$

yielding

$$\gamma_{2n+1}\,\lambda_{n-1,\nu}=\gamma_{2\nu+2}\,\theta_{n,\nu+1},\qquad n\geqslant 1\,,\quad 0\leqslant\nu\leqslant n, \tag{3.7.11}$$

since $\{R_n\}_{n\in\mathbb{N}}$ forms an independent sequence. Combining the relations (3.7.10) with $\nu$ replaced by $\nu+1$ and (3.7.11) with $n+1$ instead of $n$, we get

$$\lambda_{n+1,\nu+1}=\frac{\gamma_{2n+4}\ \gamma_{2n+3}}{\gamma_{2\nu+3}\ \gamma_{2\nu+2}}\,\lambda_{n,\nu},\quad 0\leqslant\nu\leqslant n. \tag{3.7.12}$$

Proceeding by finite induction, it is easy to deduce

$$\lambda_{n+1,\nu+1}=\left\{\prod_{\tau=0}^{2\nu+1}\frac{\gamma_{2n-2\nu+\tau+3}}{\gamma_{\tau+2}}\right\}\,\lambda_{n-\nu,0},\quad 0\leqslant\nu\leqslant n, \tag{3.7.13}$$

By virtue of (3.7.9), we are able to write

$$\lambda_{n,\nu}=\frac{1}{2\nu+1}\binom{2n+2}{2\nu}\frac{\big(2+\frac{\varepsilon}{2}\big)_{2n+1}}{\big(2+\frac{\varepsilon}{2}\big)_{2\nu}\big(2+\frac{\varepsilon}{2}\big)_{2(n-\nu)+1}}\,\lambda_{n-\nu,0}\,,\quad 1\leqslant\nu\leqslant n.$$

This last equality is identically verified when we consider the pair $(n,\nu)$ to take values on the set $\{(0,0),(1,0)\}$, so it is admissible to write:

$$\lambda_{n,\nu}=\frac{1}{2\nu+1}\binom{2n+2}{2\nu}\frac{\big(2+\frac{\varepsilon}{2}\big)_{2n+1}}{\big(2+\frac{\varepsilon}{2}\big)_{2\nu}\big(2+\frac{\varepsilon}{2}\big)_{2(n-\nu)+1}}\,\lambda_{n-\nu,0}\,,\quad 0\leqslant\nu\leqslant n. \tag{3.7.14}$$

Based on lemma 3.7.2, the determination of the coefficients $\lambda_{n-\nu,0}$ will be carried out. The particular choice $n=0$ in (3.7.5)-(3.7.6) and on account of (3.7.9), respectively, provides

$$\lambda_{0,0} = -2\left(2+\frac{\varepsilon}{2}\right) \quad , \quad \theta_{0,0} = -\left(1+\frac{\varepsilon}{2}\right). \tag{3.7.15}$$

From (3.7.10)-(3.7.11), the two following identities $\gamma_{2n+2}\theta_{n,0} = \gamma_1\lambda_{n,0}$ and $\gamma_{2n+3}\lambda_{n,0} = \gamma_2\theta_{n+1,1}$ hold. Thus, when $\nu=0$, the relations (3.7.7)-(3.7.8) given in Lemma 3.7.2 become like

$$\begin{cases} \theta_{n+1,0} = \displaystyle\sum_{\mu=0}^{n} \lambda_{n,\mu}\theta_{\mu,0}\,\beta_{2\mu+1}, \\ \lambda_{n+1,0} = \theta_{n+1,0} + \displaystyle\sum_{\mu=0}^{n} \theta_{n+1,\mu+1}\,\lambda_{\mu,0}\,\beta_{2\mu+2}, \quad n\in\mathbb{N}. \end{cases} \tag{3.7.16}$$

On account of (3.7.10) and (3.7.11), we may transform (3.7.16) into

$$\begin{cases} \dfrac{1}{\gamma_{2n+4}}\,\lambda_{n+1,0} = \displaystyle\sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\,\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+1} \\ \lambda_{n+1,0} = \dfrac{\gamma_1}{\gamma_{2n+4}}\,\lambda_{n+1,0} + \gamma_{2n+3}\displaystyle\sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\,\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+2}, \quad n\in\mathbb{N}. \end{cases} \tag{3.7.17}$$

Since, $\beta_{2\mu+2} = \beta_{2\mu+1}+2$, for $\mu \geqslant 0$, it follows

$$\sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\,\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+2} = 2\sum_{\mu=0}^{n}\left(\frac{\lambda_{n,\mu}\,\lambda_{\mu,0}}{\gamma_{2\mu+2}}\right) + \sum_{\mu=0}^{n}\left(\frac{\lambda_{n,\mu}\,\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+1}\right), \quad n\in\mathbb{N}.$$

Therefore, from (3.7.17) we derive

$$\lambda_{n+1,0} = \frac{\gamma_1}{\gamma_{2n+4}}\,\lambda_{n+1,0} + \frac{\gamma_{2n+3}}{\gamma_{2n+4}}\lambda_{n+1,0} + 2\gamma_{2n+3}\sum_{\mu=0}^{n}\frac{\lambda_{n,\mu}\,\lambda_{\mu,0}}{\gamma_{2\mu+2}}, \quad n\in\mathbb{N}, \tag{3.7.18}$$

which, on account of (3.7.9), may be written like

$$\lambda_{n+1,0} = (n+2)\,\left(2n+3+\tfrac{\varepsilon}{2}\right)\,\left(2n+4+\tfrac{\varepsilon}{2}\right)\sum_{\mu=0}^{n}\frac{\lambda_{n,\mu}\,\lambda_{\mu,0}}{(\mu+1)\,\left(2\mu+2+\frac{\varepsilon}{2}\right)}, \quad n\in\mathbb{N}. \tag{3.7.19}$$

Now, considering (3.7.14), the relation (3.7.19) becomes

$$\begin{aligned} \lambda_{n+1,0} &= (n+2)\,\left(2+\tfrac{\varepsilon}{2}\right)_{2n+3} \\ &\cdot \sum_{\mu=0}^{n}\left\{\binom{2n+2}{2\mu}\frac{\lambda_{n-\mu,0}\,\lambda_{\mu,0}}{(2\mu+1)(\mu+1)\left(2+\frac{\varepsilon}{2}\right)_{2\mu+1}\left(2+\frac{\varepsilon}{2}\right)_{2(n-\mu)+1}}\right\} \end{aligned} \tag{3.7.20}$$

and holds for all the integeres $n \in \mathbb{N}$. Proceeding by finite induction, we infer there is a set of positive integers $\{\chi_n\}_{n\in\mathbb{N}}$, not depending on the parameter $\varepsilon$, realising the equality

$$\lambda_{n,0} = (-1)^{n+1}\, 2^{2n+1}\, \chi_n\, \left(2+\tfrac{\varepsilon}{2}\right)_{2n+1}, \qquad n \in \mathbb{N}. \tag{3.7.21}$$

Indeed, on account of (3.7.15), $\chi_0 = 1$, and, under the assumption, from the relation (3.7.20) we get

$$\lambda_{n+1,0} = (n+2)\,(-1)^n\, 2^{2n+2}\, \left(2+\tfrac{\varepsilon}{2}\right)_{2n+3} \sum_{\mu=0}^{n} \left\{ \binom{2n+2}{2\mu} \frac{\chi_{n-\mu}\,\chi_\mu}{(2\mu+1)(\mu+1)} \right\},\ n \in \mathbb{N}.$$

Since the integers $\chi_n$, $n \in \mathbb{N}$, do not depend on $\varepsilon$, they are necessarily related by the equality

$$\chi_{n+1} = \frac{n+2}{2} \sum_{\mu=0}^{n} \binom{2n+2}{2\mu} \frac{\chi_{n-\mu}\,\chi_\mu}{(2\mu+1)(\mu+1)}, \qquad n \in \mathbb{N}, \tag{3.7.22}$$

or, equivalently,

$$\frac{\chi_{n+1}}{(2n+4)!} = \frac{1}{2n+3} \sum_{\mu=0}^{n} \frac{\chi_{n-\mu}}{(2n-2\mu+2)!}\,\frac{\chi_\mu}{(2\mu+2)!}, \qquad n \in \mathbb{N}. \tag{3.7.23}$$

Suppose there is an analytic function $L$ defined on an open set of $\mathbb{C}$ such that $L(z) = \sum_{n\in\mathbb{N}} \frac{\chi_n}{(2n+2)!}\, z^n$. Based upon the relation (3.7.23), $L(z)$ is a solution of the differential equation

$$\left(z\, L(z^2)\right)' = \Lambda_0 + \frac{1}{2}\left(z\, L(z^2)\right)^2 .$$

Therefore, because $\chi_0 = 1$, we trivially conclude: $z\, L\left(z^2\right) = \tan\left(\frac{z}{2}\right)$. Following, for example, [37, 105] and denoting by $\mathfrak{G}_{2n}$ the *unsigned Genocchi numbers*, it is possible to write

$$\tan\left(\frac{z}{2}\right) = \sum_{n\in\mathbb{N}} \mathfrak{G}_{2n+2}\, \frac{z^{2n+1}}{(2n+2)!}$$

whence we have $\chi_n = \mathfrak{G}_{2n+2}$ and (3.7.21) becomes

$$\lambda_{n,0} = (-1)^{n+1}\, 2^{2n+1}\, \mathfrak{G}_{2n+2}\, \left(2+\tfrac{\varepsilon}{2}\right)_{2n+1}, \qquad n \in \mathbb{N}.$$

Inserting in (3.7.14), this last equality with $n-\mu$ instead of $n$, we obtain (3.7.3) and, on account of (3.7.10), we get (3.7.4). □

The unsigned Genocchi numbers are directly related to the much more famous *Bernoulli numbers* $\mathfrak{B}_n$ via $\mathfrak{G}_{2n} = 2(1-2^{2n})\mathfrak{B}_{2n}$, where $\mathfrak{B}_n$ are defined by [37, 105]

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + \sum_{n\geqslant 1} (-1)^{n+1}\, \mathfrak{B}_{2n}\, \frac{z^{2n}}{(2n)!} . \tag{3.7.24}$$

## 3.8 Quadratic Decomposition of the $q$-Appell polynomial sequences

Now, we direct our attention toward the **q-Appell polynomial sequences**, or in order to be more closed to definition 3.0.6, we shall call them also as the $\mathbf{H_q}$**-Appell polynomial sequences**, where $H_q$ represents the operator defined as follows

$$\big(H_q f\big)(x) := \frac{f(qx) - f(x)}{(q-1)x}\,, \quad f \in \mathcal{P},$$

where $q$ belongs to the set $\widetilde{\mathbb{C}} := \mathbb{C} - \bigcup_{n\in\mathbb{N}} U_n$, with

$$U_n = \begin{cases} \{0\} & , \quad n = 0 \\ \{z \in \mathbb{C} : z^n = 1\} & , \quad n \geqslant 1. \end{cases}$$

Equivalently, recalling the definition of the operators $h_q$ and $\vartheta_0$ in (1.1.6) and (1.1.7) (p. 23), we may also write:

$$H_q = \frac{1}{q-1}\,\vartheta_0 \circ (h_q - \mathrm{I}_{\mathcal{P}})\,, \tag{3.8.1}$$

where $\mathrm{I}_{\mathcal{P}}$ represents the identity operator in $\mathcal{P}$. The operator $H_q$ is commonly called as the $q$-**derivative operator** or also as "*$q$-divided difference operator*" and is frequently denoted as $\mathcal{D}_q$. Here, we follow the notation suggested by Khériji and Maroni [59], which was motivated by the fact that the $q$-derivative is a part of what are now called *Hahn's operators*, after Hahn's work in 1949 [54]. We can define the $q$-derivative operator $H_q$ on $\mathcal{P}'$ as minus the transpose of the $q$-derivative operator on $\mathcal{P}$, that is $H_q := -{}^tH_q$, so that

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle\,, \quad f \in \mathcal{P},\ u \in \mathcal{P}',$$

and we have $H_q$ defined on $\mathcal{P}$ and $\mathcal{P}'$ leaving out a slight abuse of notation without consequence. In particular, this yields

$$\big(H_q u\big)_n = -[n]_q\,(u)_{n-1},\ n \geqslant 0,$$

with the convention $(u)_{-1} = 0$ and

$$[n]_q := \frac{q^n - 1}{q-1}, \quad n \in \mathbb{N}.$$

Next we formally list some properties of this operator $H_q$, either on $\mathcal{P}$ or on $\mathcal{P}'$, relevant for the sequel:

**Lemma 3.8.1.** *[59, 84] The following properties hold*

$$\big(H_q f_1 f_2\big)(x) = \big(h_q f_1\big)(x)\ \big(H_q f_2\big)(x) + f_2(x)\ \big(H_q f_1\big)(x), \quad f_1, f_2 \in \mathcal{P}, \tag{3.8.2}$$

$$\begin{aligned}\big(H_q f_1 f_2\big)(x) = f_1(x)\ \big(H_q f_2\big)(x) + f_2(x)\ \big(H_q f_1\big)(x) \\ +(q-1)\,x\,\big(H_q f_1\big)(x)\big(H_q f_2\big)(x), \quad f_1, f_2 \in \mathcal{P},\end{aligned} \tag{3.8.3}$$

$$\big(h_a f_1 f_2\big)(x) = \big(h_a f_1\big)(x)\ \big(h_a f_2\big)(x), \quad f_1, f_2 \in \mathcal{P},\ a \in \mathbb{C} - \{0\}, \tag{3.8.4}$$

$$h_a\big(gu\big) = \big(h_{a^{-1}} g\big)\ \big(h_a u\big), \quad g \in \mathcal{P}, u \in \mathcal{P}'\ a \in \mathbb{C} - \{0\}, \tag{3.8.5}$$

$$H_q\big(gu\big) = g\ H_q u + \Big(H_{q^{-1}} g\Big) h_q u, \quad g \in \mathcal{P}, u \in \mathcal{P}' \tag{3.8.6}$$

$$H_q\big(gu\big) = \big(h_{q^{-1}} g\big)\ H_q u + q^{-1}\Big(H_{q^{-1}} g\Big) u, \quad g \in \mathcal{P}, u \in \mathcal{P}' \tag{3.8.7}$$

$$H_q \circ h_{q^{-1}} = q^{-1} H_{q^{-1}} \quad \textit{in } \mathcal{P} \tag{3.8.8}$$

$$h_{q^{-1}} \circ H_q = H_{q^{-1}} \quad \textit{in } \mathcal{P} \tag{3.8.9}$$

$$H_q \circ h_a = a\, h_a \circ H_q \quad \textit{in } \mathcal{P} \quad (\textit{with } a \in \mathbb{C} - \{0\}), \tag{3.8.10}$$

$$H_q \circ H_{q^{-1}} = q^{-1}\ H_{q^{-1}} \circ H_q \quad \textit{in } \mathcal{P} \tag{3.8.11}$$

$$H_q \circ h_{q^{-1}} = H_{q^{-1}} \quad \textit{in } \mathcal{P}' \tag{3.8.12}$$

$$h_{q^{-1}} \circ H_q = q^{-1}\ H_{q^{-1}} \quad \textit{in } \mathcal{P}' \tag{3.8.13}$$

$$H_q \circ h_a = a^{-1}\ h_a \circ H_q \quad \textit{in } \mathcal{P}' \quad (\textit{with } a \in \mathbb{C} - \{0\}), \tag{3.8.14}$$

$$H_q \circ H_{q^{-1}} = q\ H_{q^{-1}} \circ H_q \quad \textit{in } \mathcal{P}' \tag{3.8.15}$$

$$\Big(H_q\big(h_{q^{-1}} f_1\big)\ f_2\Big)(x) = f_1(x)\big(H_q f_2\big)(x) + q^{-1} f_2(x)\big(H_{q^{-1}} f_1\big)(x), \quad f_1, f_2 \in \mathcal{P}, \tag{3.8.16}$$

*The operator $H_q$ is injective in $\mathcal{P}'$.* (3.8.17)

Clearly $H_q$ is a lowering operator. In accordance with (3.0.3), from a given MPS $\{B_n\}_{n\in\mathbb{N}}$ we construct the sequence of $q$-derivatives $\{B_n^{[1]}(\cdot;q) := B_n^{[1]}(\cdot;H_q)\}_{n\in\mathbb{N}}$ as follows

$$B_n^{[1]}(x;q) := \frac{1}{[n+1]_q}\ \big(H_q B_{n+1}\big)(x), n \in \mathbb{N}. \tag{3.8.18}$$

Naturally, $\{B_n^{[1]}(\cdot;q)\}_{n\in\mathbb{N}}$ is a MPS. Let us denote, as usual, by $\{u_n\}_{n\in\mathbb{N}}$ the dual sequence associated to $\{B_n\}_{n\in\mathbb{N}}$ and by $\{u_n^{[1]}(q)\}_{n\in\mathbb{N}}$ the one of $\{B_n^{[1]}(\cdot;q)\}_{n\in\mathbb{N}}$. As a consequence of lemma 1.3.1, it comes out the relation (the proof of this result may be followed in [59]):

$$H_q\Big(u_n^{[1]}(q)\Big) = -[n+1]_q\, u_{n+1}, \quad n \in \mathbb{N}. \tag{3.8.19}$$

Following definition 3.0.6 and (3.8.18), the MPS $\{B_n\}_{n\in\mathbb{N}}$ is a **q-Appell sequence** whenever $B_n(\cdot) = B_n^{[1]}(\cdot,q)$ , $n \in \mathbb{N}$. The dual sequence of a given MPS is uniquely determined, therefore on account of (3.8.19), the elements of its dual sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfy

$$u_{n+1} = -\frac{1}{[n+1]_q}\,(H_q u_n)\,, \quad n \in \mathbb{N}. \tag{3.8.20}$$

**Proposition 3.8.2.** *The elements of the dual sequence $\{u_n\}_{n\in\mathbb{N}}$ of an $H_q$-Appell sequence $\{B_n\}_{n\in\mathbb{N}}$ may be expressed by*

$$u_n = \frac{(-1)^n}{[n+1]_q!}\,\left({H_q}^n\, u_0\right)\,, \quad n \in \mathbb{N}, \tag{3.8.21}$$

*where the symbol $[z]_q! = [z]_q\,[z-1]_q\ldots[1]_q$ represents the q-factorial of the integer $z$.*

*Proof.* The condition is necessary. The dual sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies (3.8.20) for any integer $n \in \mathbb{N}$. In particular, considering $n = 0$, we obtain

$$u_1 = -H_q(u_0)\,. \tag{3.8.22}$$

By recurrence, we get (3.8.21).

Conversely, the relation (3.8.20) provides (3.8.20), and when compared to (3.8.19) leads to the equality

$$H_q\left(u_n^{[1]}(q)\right) = H_q\big(u_n\big),\ n \in \mathbb{N}\,.$$

The lowering operator $H_q$ satisfies $H_q(\mathcal{P}) = \mathcal{P}$, and therefore $H_q$ is one-to-one on $\mathcal{P}'$. Consequently, we get $u_n^{[1]}(q) = u_n\,,\ n \in \mathbb{N}$, whence the expected result. □

Among all the possible $H_q$-Appell sequences, there is a particular group that ought to have a special attention: the orthogonal ones.

**Proposition 3.8.3.** *The unique $H_q$-Appell orthogonal polynomial sequences are the $q$-polynomials of Al-Salam and Carlitz [5], up to a linear transformation, and the recurrence coefficients $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$ associated to the corresponding second order recurrence relation are given by*

$$\beta_n = \beta_0\, q^n\,, \quad n \in \mathbb{N},$$
$$\gamma_{n+1} = q^n [n+1]_q\, \gamma_1\,, \quad n \in \mathbb{N},$$

*where $\beta_0$ and $\gamma_1 \neq 0$ are two arbitrary constants.*

This result is not new and it may be followed in the work of Khériji and Maroni [59] when the authors were studying all the MOPS whose $q$-derivative sequence $\{B_n^{[1]}(\cdot;q)\}_{n\in\mathbb{N}}$ was also MOPS. From a combinatorial perspective, they are also interpreted as the $q$-analogues of the *Charlier polynomials*.

*Proof.* Suppose $\{B_n\}_{n\in\mathbb{N}}$ is $q$-Appell MOPS. Since the orthogonality of $\{B_n\}_{n\in\mathbb{N}}$ implies the elements of its corresponding dual sequence $\{u_n\}_{n\in\mathbb{N}}$ to be given by (1.4.2), then the relation (3.8.20) may be transformed into the following one

$$H_q\big(B_n\, u_0\big) = -\lambda_n\, B_{n+1}\, u_0, \quad n \in \mathbb{N}, \tag{3.8.23}$$

where

$$\lambda_n = \frac{[n+1]_q}{\gamma_{n+1}}, \quad n \in \mathbb{N}. \tag{3.8.24}$$

Taking $n=0$ in (3.8.23), we obtain

$$H_q(u_0) + {\gamma_1}^{-1}\, B_1\, u_0 = 0. \tag{3.8.25}$$

The equalities in (3.8.23) may also be expressed, due to (3.8.7), like

$$\big(h_{q^{-1}} B_n\big)\big(H_q u_0\big) + q^{-1}\left(H_{q^{-1}} B_n\right)\; u_0 = -\lambda_n\, B_{n+1}\, u_0, \quad n \in \mathbb{N},$$

yielding, after (3.8.25),

$$\big\{ -{\gamma_1}^{-1}\, B_1\; \big(h_{q^{-1}} B_n\big) + q^{-1}\left(H_{q^{-1} B_n}\right) \big\}\; u_0 = -\lambda_n\, B_{n+1} u_0, \quad n \in \mathbb{N}.$$

The regularity of $u_0$ permits to obtain from the previous relations

$$\big\{ -{\gamma_1}^{-1}\, B_1\; \big(h_{q^{-1}} B_n\big) + q^{-1}\left(H_{q^{-1}} B_n\right) \big\} = -\lambda_n\, B_{n+1}, \quad n \in \mathbb{N}.$$

Operating with $h_q$ on both sides of the foregoing equalities, we derive, on account of (3.8.4),

$$\big\{ -{\gamma_1}^{-1}\; (h_q B_1)\; B_n + q^{-1}\left(h_q \circ H_{q^{-1}} B_n\right) \big\} = -\lambda_n\; (h_q B_{n+1})\,, \quad n \in \mathbb{N},$$

which, due to (3.8.9) with $q^{-1}$ instead of $q$, may be written as

$$-{\gamma_1}^{-1}\; (h_q B_1)\; B_n + q^{-1}\left(H_q B_n\right) = -\lambda_n\; (h_q B_{n+1})\,, \quad n \in \mathbb{N},$$

or, because $h_q = (q-1)\, x\, H_q + \mathrm{I}_{\mathcal{P}}$ on $\mathcal{P}$, also as

$$-{\gamma_1}^{-1}\; (h_q B_1)\; B_n + q^{-1}\left(H_q B_n\right) = -\lambda_n\, \big\{(q-1)\, x \left(H_q B_{n+1}\right) + B_{n+1}\big\}, \quad n \in \mathbb{N}. \tag{3.8.26}$$

The $H_q$-Appell character of $\{B_n\}_{n\geqslant}$ provides $H_q B_{n+1} = [n+1]_q B_n$, $n \in \mathbb{N}$, so (3.8.26) with $n$ replaced by $n+1$ becomes

$$-{\gamma_1}^{-1}\; (h_q B_1)\; B_{n+1} + q^{-1}[n+1]_q\, B_n = -\lambda_{n+1}\, \big\{(q-1)\; x\; [n+2]_q\, B_{n+1} + B_{n+2}\big\}, \quad n \in \mathbb{N},$$

and reordering the terms we finally get the second order relation:

$$B_{n+2} = \left\{ \frac{\lambda_{n+1}\; (q-1)\; [n+2]_q\; x\; - \gamma_1^{-1}\; (h_q B_1)}{-\lambda_{n+1}} \right\}\; B_{n+1} + \frac{q^{-1}[n+1]_q}{-\lambda_{n+1}}\; B_n, \quad n \in \mathbb{N},$$

*i.e.*,

$$B_{n+2}=\left\{\frac{\lambda_{n+1}\,(q-1)\,[n+2]_q\ -\gamma_1^{-1}q}{-\lambda_{n+1}}\,x\ -\beta_0\frac{\gamma_1^{-1}}{\lambda_{n+1}}\right\}\,B_{n+1}-\frac{q^{-1}[n+1]_q}{\lambda_{n+1}}\,B_n,\quad n\in\mathbb{N}. \tag{3.8.27}$$

The orthogonality of $\{B_n\}_{n\in\mathbb{N}}$ assures the existence of a unique set of recurrence coefficients $(\beta_n,\gamma_{n+1})_{n\in\mathbb{N}}$ such that

$$\begin{array}{l} B_0(x)=1\quad ;\quad B_1(x)=x-\beta_0\\ B_{n+2}(x)=(x-\beta_{n+1})B_{n+1}(x)-\gamma_{n+1}B_n(x)\ ,\quad n\in\mathbb{N}, \end{array}$$

consequently, upon the comparison with (3.8.27), we obtain the system

$$\begin{cases} -\lambda_{n+1}=\lambda_{n+1}\,(q-1)\,[n+2]_q\ -\gamma_1^{-1}q\ ,\quad n\in\mathbb{N},\\ \beta_{n+1}=\beta_0\dfrac{\gamma_1^{-1}}{\lambda_{n+1}}\ ,\quad n\in\mathbb{N},\\ \gamma_{n+1}=\dfrac{q^{-1}[n+1]_q}{\lambda_{n+1}}\ ,\quad n\in\mathbb{N}, \end{cases}$$

*i.e.* ,

$$\begin{cases} \big\{1+(q-1)\,[n+2]_q\big\}\lambda_{n+1}=\ \gamma_1^{-1}q\ ,\quad n\in\mathbb{N},\\ \beta_{n+1}=\beta_0\dfrac{\gamma_1^{-1}}{\lambda_{n+1}}\ ,\quad n\in\mathbb{N},\\ \lambda_{n+1}=\dfrac{q^{-1}[n+1]_q}{\gamma_{n+1}}\ ,\quad n\in\mathbb{N}, \end{cases}$$

*i.e.* ,

$$\begin{cases} \gamma_{n+1}=\big\{1+(q-1)\,[n+2]_q\big\}\dfrac{q^{-1}[n+1]_q}{\gamma_1^{-1}q}\ ,\quad n\in\mathbb{N},\\ \beta_{n+1}=\beta_0\ \dfrac{\gamma_1^{-1}\ \gamma_{n+1}}{q^{-1}[n+1]_q}\ ,\quad n\in\mathbb{N},\\ \lambda_{n+1}=\dfrac{q^{-1}[n+1]_q}{\gamma_{n+1}}\ ,\quad n\in\mathbb{N}, \end{cases}$$

Since $(q-1)[n]_q=q^n-1$, we then have

$$\begin{cases} \gamma_{n+1}=q^n\ [n+1]_q\,\gamma_1\ ,\quad n\in\mathbb{N},\\ \beta_{n+1}=\beta_0\ q^{n+1}\ ,\quad n\in\mathbb{N},\\ \lambda_{n+1}=\dfrac{1}{q^{n+1}\ \gamma_1}\ ,\quad n\in\mathbb{N}. \end{cases}$$

As matter of fact the third condition of this last system is redundant once, recalling (3.8.24), it provides the equality $\gamma_{n+2}=q^{n+1}\ [n+2]_q\ \gamma_1\ ,\ n\in\mathbb{N}$. □

Given a MPS $\{B_n\}_{n\in\mathbb{N}}$ there exist two MPS $\{P_n\}_{n\in\mathbb{N}}$, $\{R_n\}_{n\in\mathbb{N}}$ and two sequences of polynomials $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ permitting the description according to (3.0.1)-(3.0.2). Under the assumption of $\{B_n\}_{n\in\mathbb{N}}$ being $q$-Appell we are interested in finding useful information about those four associated sequences. For instance, do those sequences play a role of Appell sequences with respect to another lowering $q$-differential operator?

The answer to such questions require the knowledge of some properties about $H_q$ not listed in (3.8.2)-(3.8.17), namely:

$$\Big(H_q p(\xi^2)\Big)(x) = x\,\Big\{q\,\big(H_q p\big)(q\,x^2) + \big(H_q p\big)(x^2)\Big\},\quad \forall p\in\mathcal{P} \tag{3.8.28}$$

or, equivalently,

$$H_q\circ\sigma \;=\; x\,\sigma\circ H_q\circ(\,\mathrm{I}_{\mathcal{P}}+h_q\,)\quad \text{in } \mathcal{P}\,. \tag{3.8.29}$$

In addition, from (3.8.2) it follows $\big(H_q\xi f(\xi)\big)(x) = q\,x\big(H_q f\big)(x) + f(x)$ , that is

$$H_q\,x = q\,x\,H_q + \mathrm{I}_{\mathcal{P}}\quad \text{in } \mathcal{P}\,, \tag{3.8.30}$$

therefore, due to (3.8.30) and then (3.8.29), we derive:

$$\begin{aligned} H_q\,x\;\circ\sigma &= q\,x\,\Big(H_q\;\circ\;\sigma\Big) + \sigma \\ &= q\,x\,\Big(x\,\sigma\circ H_q + x\,\sigma\circ H_q\circ h_q\Big) + \sigma \\ &= \sigma\Big\{q\,x\,H_q\Big(h_q + \mathrm{I}_{\mathcal{P}}\Big) + \mathrm{I}_{\mathcal{P}}\Big\} \end{aligned}$$

whence

$$H_q\,x\;\circ\sigma = \sigma\Big\{q\,x\,H_q\Big(h_q + \mathrm{I}_{\mathcal{P}}\Big) + \mathrm{I}_{\mathcal{P}}\Big\}\quad \text{in } \mathcal{P}\,. \tag{3.8.31}$$

or, equivalently,

$$\Big(H_q\,\xi\,p(\xi^2)\Big)(x) = qx^2\,\Big(H_q p\Big)(qx^2) + qx^2 p(x^2) + p(x^2),\quad \forall p\in\mathcal{P}\,.$$

**Lemma 3.8.4.** *Consider the quadratic decomposition of the MPS $\{B_n\}_{n\in\mathbb{N}}$ according to* (3.0.1)-(3.0.2). *If $\{B_n\}_{n\in\mathbb{N}}$ is $q$-Appell then the sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ are Appell sequences with respect to another $q$-differential operator. Moreover,*

$$R_n(x) = \frac{1}{[2n+2]_q\,[2n+3]_q}\Big(\mathcal{M}_q^{(+1)}R_{n+1}\Big)(x)\,,\quad n\in\mathbb{N}, \tag{3.8.32}$$

$$P_n(x) = \frac{1}{[2n+1]_q\,[2n+2]_q}\Big(\mathcal{M}_q^{(-1)}P_{n+1}\Big)(x)\,,\quad n\in\mathbb{N}. \tag{3.8.33}$$

$$b_n(x) = \frac{1}{[2n+2]_q\,[2n+3]_q}\Big(\mathcal{M}_q^{(-1)}b_{n+1}\Big)(x)\,,\quad n\in\mathbb{N}, \tag{3.8.34}$$

$$a_n(x) = \frac{1}{[2n+3]_q\,[2n+4]_q}\Big(\mathcal{M}_q^{(+1)}a_{n+1}\Big)(x)\,,\quad n\in\mathbb{N}. \tag{3.8.35}$$

*with*

$$\begin{aligned}\mathcal{M}_q^{(\varepsilon)} \;&=\; q^{\varepsilon}\Big\{(q-1)^2\, H_q\big(xH_q\big)^3 + 4(q-1)\, H_q\big(xH_q\big)^2 + 5\, H_q\, x\, H_q\Big\}\\ &\quad - H_q\, x\, H_q + \{(\varepsilon+1) + q^{\varepsilon}(\varepsilon-1)\}H_q\end{aligned} \tag{3.8.36}$$

*Proof.* Representing by $\{B_n\}_{n\in\mathbb{N}}$ a $q$-Appell sequence, we proceed to its quadratic decompositionin accordance with (3.0.1)-(3.0.2). Operating with $H_q$ on both sides of (3.0.1), after replacing $n$ by $n+1$, and on (3.0.2), we respectively obtain

$$[2n+2]_q\, B_{2n+1}(x) = \Big(H_q P_{n+1}(\xi^2)\Big)(x) + \Big(H_q \xi\, a_n(\xi^2)\Big)(x), \quad n\in\mathbb{N}, \tag{3.8.37}$$

$$[2n+1]_q B_{2n}(x) = \Big(H_q b_n(\xi^2)\Big)(x) + \Big(H_q \xi\, R_n(\xi^2)\Big)(x), \quad n\in\mathbb{N}, \tag{3.8.38}$$

since the $H_q$-Appell character of $\{B_n\}_{n\in\mathbb{N}}$ provides $\big(H_q B_{n+1}\big)(x) = [n+1]_q\, B_n(x)$, $n\in\mathbb{N}$. Equating (3.8.37) with (3.0.2), we have

$$[2n+2]_q\Big\{b_n(x^2) + x\, R_n(x^2)\Big\} = \Big(H_q P_{n+1}(\xi^2)\Big)(x) + \Big(H_q\xi\, a_n(\xi^2)\Big)(x), \quad n\in\mathbb{N}.$$

Likewise, the comparison between (3.8.38) and (3.0.1) leads to

$$[2n+1]_q\Big\{P_n(x^2) + x\, a_{n-1}(x^2)\Big\} = \Big(H_q b_n(\xi^2)\Big)(x) + \Big(H_q\xi\, R_n(\xi^2)\Big)(x)\,, \quad n\in\mathbb{N}.$$

On account of (3.8.29) and (3.8.31) the previous two relations become respectively as follows:

$$\begin{aligned}[2n+2]_q\Big\{\sigma\, b_n(x) + x\,\sigma\, R_n(x)\Big\} &= \Big(x\,\sigma\circ H_q\circ(\,\mathrm{I}_{\mathcal{P}} + h_q\,)\,P_{n+1}\Big)(x)\\ &\quad + \Big(\sigma\Big\{q\,x\,H_q\Big(h_q + \mathrm{I}_{\mathcal{P}}\Big) + \mathrm{I}_{\mathcal{P}}\Big\}\,a_n\Big)\,(x)\,, \quad n\in\mathbb{N},\end{aligned} \tag{3.8.39}$$

$$\begin{aligned}[2n+1]_q\Big\{\sigma\, P_n(x) + x\,\sigma\, a_{n-1}(x)\Big\} &= \Big(x\,\sigma\circ H_q\circ(\,\mathrm{I}_{\mathcal{P}} + h_q\,)\,b_n\Big)(x)\\ &\quad + \Big(\sigma\Big\{q\,x\,H_q\Big(h_q + \mathrm{I}_{\mathcal{P}}\Big) + \mathrm{I}_{\mathcal{P}}\Big\}\,R_n\Big)\,(x)\,, \quad n\in\mathbb{N}.\end{aligned} \tag{3.8.40}$$

Equating the even and odd terms in (3.8.39) and in (3.8.40), we respectively have:

$$[2n+2]_q\, R_n(x) = \Big(H_q\circ(\,\mathrm{I}_{\mathcal{P}} + h_q\,)\,P_{n+1}\Big)(x)\,, \quad n\in\mathbb{N}, \tag{3.8.41}$$

$$[2n+2]_q\, b_n(x) = \Big(\Big(q\,x\,H_q\big(h_q + \mathrm{I}_{\mathcal{P}}\big) + \mathrm{I}_{\mathcal{P}}\Big)\,a_n\Big)\,(x)\,, \quad n\in\mathbb{N}, \tag{3.8.42}$$

$$[2n+1]_q\, P_n(x) = \Big(\Big(q\,x\,H_q\big(h_q + \mathrm{I}_{\mathcal{P}}\big) + \mathrm{I}_{\mathcal{P}}\Big)\,R_n\Big)\,(x)\,, \quad n\in\mathbb{N}, \tag{3.8.43}$$

$$[2n+1]_q\, a_{n-1}(x) = \Big(H_q\circ(\,\mathrm{I}_{\mathcal{P}} + h_q\,)\,b_n\Big)(x)\,, \quad n\geqslant 1. \tag{3.8.44}$$

The relations (3.8.41)-(3.8.43), provide

$$[2n+2]_q\ [2n+3]_q\ R_n(x) = \Big(\mathcal{M}_q^{(+1)} R_{n+1}\Big)(x)\,,\quad n\in\mathbb{N}, \tag{3.8.45}$$

$$[2n+1]_q\ [2n+2]_q\ P_n(x) = \Big(\mathcal{M}_q^{(-1)} P_{n+1}\Big)(x)\,,\quad n\in\mathbb{N}. \tag{3.8.46}$$

with

$$\mathcal{M}_q^{(+1)} := \Big(H_q\circ(\,\mathrm{I}_{\mathcal{P}} + h_q\,)\Big)\circ\Big(q\ x\ H_q\big(h_q+\mathrm{I}_{\mathcal{P}}\big)+\mathrm{I}_{\mathcal{P}}\Big) \tag{3.8.47}$$

$$\mathcal{M}_q^{(-1)} := \Big(q\ x\ H_q\big(h_q+\mathrm{I}_{\mathcal{P}}\big)+\mathrm{I}_{\mathcal{P}}\Big)\circ\Big(H_q\circ(\,\mathrm{I}_{\mathcal{P}} + h_q\,)\Big) \tag{3.8.48}$$

Analogously, based on (3.8.42) and (3.8.44), we conclude:

$$[2n+2]_q\ [2n+3]_q\ b_n(x) = \Big(\mathcal{M}_q^{(-1)} b_{n+1}\Big)(x)\,,\quad n\in\mathbb{N}, \tag{3.8.49}$$

$$[2n+1]_q\ [2n+2]_q\ a_{n-1}(x) = \Big(\mathcal{M}_q^{(+1)} a_{n}\Big)(x)\,,\quad n\geqslant 1. \tag{3.8.50}$$

If we set

$$F_q := H_q\circ(\,\mathrm{I}_{\mathcal{P}} + h_q\,)\ , \tag{3.8.51}$$

then the operators $\mathcal{M}_q^{(+1)}$ and $\mathcal{M}_q^{(-1)}$ become respectively like:

$$\mathcal{M}_q^{(+1)} = q\ F_q x F_q + F_q \tag{3.8.52}$$

$$\mathcal{M}_q^{(-1)} = qx\ F_q\circ F_q + F_q \tag{3.8.53}$$

Since $h_q = (q-1)xH_q + \mathrm{I}_{\mathcal{P}}$, we have

$$F_q = (q-1)\ H_q\ x\ H_q + 2\ H_q \tag{3.8.54}$$

therefore, from (3.8.30) it follows

$$\begin{aligned} x\ F_q &= (q-1)\ x\ H_q\ x\ H_q + 2\ x\ H_q \\ &= (q-1)\ q^{-2}\Big(H_q\ x - \mathrm{I}_{\mathcal{P}}\Big)\Big(H_q\ x - \mathrm{I}_{\mathcal{P}}\Big) + 2q^{-1}\ \Big(H_q\ x - \mathrm{I}_{\mathcal{P}}\Big) \\ &= q^{-2}(q-1)\ \Big(H_q\ x\ H_q - 2\ H_q\ x + \mathrm{I}_{\mathcal{P}}\Big) + 2q^{-1}\ \Big(H_q\ x - \mathrm{I}_{\mathcal{P}}\Big) \\ &= q^{-2}(q-1)\ H_q\ x\ H_q + 2q^{-2}\ H_q\ x - q^{-2}(q+1)\mathrm{I}_{\mathcal{P}} \end{aligned}$$

whence, we derive

$$x\ F_q = q^{-2}\ F_q\ x - q^{-2}(q+1)\ \mathrm{I}_{\mathcal{P}}$$

and this provides

$$\begin{aligned} \mathcal{M}_q^{(+1)} &= q\ F_q\ x\ F_q + F_q \\ \mathcal{M}_q^{(-1)} &= q^{-1}\ F_q\ x\ F_q - q^{-1}(q+1)\ F_q + F_q \end{aligned}$$

i.e.,

$$\mathcal{M}_q^{(+1)} = q\, F_q\, x\, F_q + F_q$$
$$\mathcal{M}_q^{(-1)} = q^{-1}\, F_q\, x\, F_q - q^{-1}\, F_q$$

Based on the expression of $F_q$ given by (3.8.54), these operators may also be written like:

$$\begin{aligned}
\mathcal{M}_q^{(+1)} &= q \left\{(q-1)\, H_q\, x\, H_q + 2\, H_q\right\} x \left\{(q-1)\, H_q\, x\, H_q + 2\, H_q\right\} \\
&\quad + \left\{(q-1)\, H_q\, x\, H_q + 2\, H_q\right\} \\
\mathcal{M}_q^{(-1)} &= q^{-1} \left\{(q-1)\, H_q\, x\, H_q + 2\, H_q\right\} x \left\{(q-1)\, H_q\, x\, H_q + 2\, H_q\right\} \\
&\quad - q^{-1} \left\{(q-1)\, H_q\, x\, H_q + 2\, H_q\right\}
\end{aligned}$$

*i.e.* ,

$$\begin{aligned}
\mathcal{M}_q^{(+1)} &= q\,(q-1)^2\, H_q\, x\, H_q\, x\, H_q\, x\, H_q + 4\, q(q-1)\, H_q\, x\, H_q\, x\, H_q \\
&\quad + 4\, q\, H_q\, x\, H_q + (q-1)\, H_q\, x\, H_q + 2\, H_q \\
\mathcal{M}_q^{(-1)} &= q^{-1}\,(q-1)^2\, H_q\, x\, H_q\, x\, H_q\, x\, H_q + 4\, q^{-1}(q-1)\, H_q\, x\, H_q\, x\, H_q \\
&\quad + 4\, q^{-1} H_q\, x\, H_q + (q^{-1}-1)\, H_q\, x\, H_q - 2q^{-1}\, H_q
\end{aligned}$$

*i.e.* ,

$$\begin{aligned}
\mathcal{M}_q^{(+1)} &= q \left\{\, (q-1)^2\, H_q\, x\, H_q\, x\, H_q\, x\, H_q + 4\,(q-1)\, H_q\, x\, H_q\, x\, H_q + 5\;\, H_q\, x\, H_q\right\} \\
&\quad - H_q\, x\, H_q + 2\, H_q \\
\mathcal{M}_q^{(-1)} &= q^{-1} \left\{(q-1)^2\, H_q\, x\, H_q\, x\, H_q\, x\, H_q + 4\,(q-1)\, H_q\, x\, H_q\, x\, H_q + 5\, H_q\, x\, H_q\right\} \\
&\quad - H_q\, x\, H_q - 2q^{-1}\, H_q
\end{aligned}$$

Considering the $k^{\text{th}}$-power of $xH_q$, $(xH_q)^{k+1} := xH_q(xH_q)^k$, for $k \in \mathbb{N}$, with the convention $(x\, H_q)^0 := \mathrm{I}_{\mathcal{P}}$, the operator $\mathcal{M}_q^{(\varepsilon)}$ with $\varepsilon \in \{-1,+1\}$ may be represented by (3.8.36). □

The two operators $\mathcal{M}_q^{(+1)}$ and $\mathcal{M}_q^{(-1)}$ arisen with this last result are two lowering operators. Therefore, in the light of definition 3.0.6, from the obtained relations (3.8.32) and (3.8.33) we may read that the two MPS $\{P_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ are $\mathcal{M}_q^{(+1)}$-Appell and $\mathcal{M}_q^{(-1)}$-Appell sequences, respectively. Analogously to the study taken over the $\mathcal{F}_\varepsilon$-Appell and $\mathcal{G}_{\varepsilon,\mu}$-Appell sequences we envisage here a promenade to be made with the research about the $\mathcal{M}_q^{(\varepsilon)}$-Appell sequences for some complex parameter $\varepsilon$. We will leave this boulevard to be explored in a future work.

# CHAPTER 4

## Hahn's problem with respect to other operators

Hahn has described the collection of orthogonal polynomial sequences $\{B_n\}_{n\in\mathbb{N}}$ which share the property of the sequence of derivatives $\{B'_n\}_{n\in\mathbb{N}}$ being also orthogonal: the so-called classical sequences. However, one might wonder about the properties shared among all the orthogonal sequences $\{B_n\}_{n\in\mathbb{N}}$ such that the new sequence $\{\mathcal{O}\, B_n(\cdot)\}_{n\in\mathbb{N}}$, in which $\mathcal{O}$ represents either a *lowering operator* (please consult p.86) or a (linear) isomorphism in $\mathcal{P}$, is also an orthogonal sequence. In other words, upon the introduction of an operator $\mathcal{O}$ mapping $\mathcal{P}$ into itself, possessing a certain number of necessary properties, we are looking for, in Hahn's sense, all the $\mathcal{O}$-classical sequences, apropos the importance, a more formally description is next given.

**Definition 4.0.5.** A MOPS $\{B_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{O}$-classical sequence whenever the MPS $\{B_n^{[1]}(\cdot;\mathcal{O})\}_{n\in\mathbb{N}}$ is also orthogonal.

The sequence $\{B_n^{[1]}(\cdot;\mathcal{O})\}_{n\in\mathbb{N}}$ mentioned in the previous definition is defined in (3.0.3) if $\mathcal{O}$ is a lowering operator, and in the case $\mathcal{O}$ is an isomorphic operator, it is merely analogous (an example, although meaningless, is the isomorphic operator $h_a \circ \tau_b$ considered in p.27).

This problem goes back to 1949, when Hahn [54] brought into light some remarkable properties shared by all the now called $\mathcal{L}_{q,\omega}$-classical sequences, where $\mathcal{L}_{q,\omega}$ defined by $\mathcal{L}_{q,\omega} f(x) := \frac{f(qx+\omega)-f(x)}{(q-1)x+\omega}$, for real numbers $q,\omega$ and for any $f \in \mathcal{P}$. This problem gave rise to many others, specially in the field of the classical $q$-analogue polynomial sequences, which has been widely explored. The $\mathcal{O}$-classical sequences when $\mathcal{O} = \mathcal{L}_{1,\omega}$ , $\mathcal{L}_{q,0}$ are completely described in the works by Abdelkarim and Maroni [1] and by Khériji and Maroni [59], respectively. Recent researches about the so called Dunkl-classical polynomial sequences, that is, classical sequences with respect to the Dunkl [38] operator defined by $\mathcal{D} := D + \vartheta H_{-1}$ have been in discussion: see for instance Ben Cheikh and Gaied [13], Ghressi and Khériji [50]. There are many other examples which support the importance of the study of the classical sequences in this wider sense, since it brings to light desirable properties of some orthogonal sequences.

Also Krall and Sheffer [67] attempted to determine the $L_k$-classical sequences, where $L_k$ corresponds to the differential operator defined by $L_k = \sum_{j=0}^{k} a_j(x)\, D^{j+1}$ with $a_j$ representing a polynomial of degree $\leqslant j$ for $k \in \mathbb{N}$. The technical problems inherent to this issue makes such problem almost impossible to solve for any integer $k \in \mathbb{N}$, so only some properties may be obtained. Therefore, they succeeded in finding the $L$-classical sequences in some particular cases. Later on, Kwon and Yoon [68] revisited this problem using techniques that were not yet available at the time of Krall and Sheffer's approach. Again, they were able to obtain some results only for some particular choices of the integer $k$ and the polynomials $a_j$. In both works, the results obtained were based on the two works of Krall [64, 65].

In Chapter 3 of the present work, three lowering operators came up with the QD of Appell sequences, namely the two lowering operators $\mathcal{F}_\varepsilon$ and $\mathcal{G}_{\varepsilon,\mu}$ respectively given by (3.1.5) and (3.4.22). The research on the $\mathcal{F}_\varepsilon$ and $\mathcal{G}_{\varepsilon,\mu}$-classical sequences, in the light of definition 4.0.5, has already started in sections 3.3 and 3.6, when all the orthogonal sequences possessing the $\mathcal{F}_\varepsilon$ and $\mathcal{G}_{\varepsilon,\mu}$-Appell character were described, respectively. Within this framework, we are now capable of characterise all the $\mathcal{F}_\varepsilon$-classical sequences. Unfortunately the study of the $\mathcal{G}_{\varepsilon,\mu}$-classical sequences will be left to a future work for reasons to be announced later.

In the search of the $\mathcal{F}_\varepsilon$-classical sequences we could have followed the work of Kwon and Yoon [68] or the techniques of Krall and Sheffer [67], nevertheless the developments presented here will be made according to the approach presented by Maroni [78] in the characterisation of the $(D)$-classical sequences, and also used in Abdelkarim and Maroni [1], Khériji and Maroni [59], Maroni and Mejri [87] to characterise the classical sequences with respect to the operators $D_\omega := \mathcal{L}_{1,\omega}$, $H_q := \mathcal{L}_{q,0}$ and $\mathcal{I}_{q,\omega}$, respectively, with some adjustments required for technical reasons. These adjustments were rather important to the process of the characterisation of other $L_k$-classical sequences in a consistent way, but we will not perform this study in this work.

This chapter targets at the characterisation of all the $\mathcal{F}_\varepsilon$-classical sequences. However, such study will be preceded with the characterisation of classical sequences with respect to an isomorphic operator consisting on a linear first order differential operator, here denoted as $\mathcal{I}_\xi$ presented below in (4.1.2). As a matter of fact, classical sequences with respect to (other) isomorphic operators have already been expounded, see for instance, the work of Maroni and Mejri [87].

Later on, at the end of Section 4.1, we will show that the sequence of derivatives of a $\mathcal{I}_\xi$-classical sequence is indeed a $\mathcal{F}_\varepsilon$-classical sequence. Finally, in Section 4.2, we will demonstrate whether there are other $\mathcal{F}_\varepsilon$-classical sequences.

## 4.1 Example of an isomorphic operator

Before diving into the analysis of all the classical sequences with respect to differential operator $\mathcal{F}_\varepsilon$, let us analyse which are the sequences possessing the Hahn's property with respect to the linear differential operator $\mathcal{I}_\xi = Dx + \xi\,\mathbb{I}$ for some complex parameter $\xi$. Clearly, $\mathcal{I}_\xi$ is an isomorphism on $\mathcal{P}$ (and also on $\mathcal{P}'$). The problem just pointed out corresponds to the search of all the MOPS being $\mathcal{I}_\xi$-classical.

Given a MPS $\{P_n\}_{n\in\mathbb{N}}$, it is possible to construct the polynomial sequence $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$ defined through

$$P_n^{[1]}(x;\mathcal{I}_\xi) := \frac{1}{n+1+\xi}\left(\mathcal{I}_\xi P_n\right)(x)\,, \qquad n\in\mathbb{N}, \tag{4.1.1}$$

with

$$\mathcal{I}_\xi := D\,x + \xi\,\mathbb{I}_{\mathcal{P}} \tag{4.1.2}$$

where $\mathbb{I}_{\mathcal{P}}$ denotes the identity operator on $\mathcal{P}$ and $\xi$ represents a complex parameter such that

$$\xi \neq -(n+1),\ n\in\mathbb{N}\,. \tag{4.1.3}$$

Naturally, $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$ is also a MPS.

Please note that, for the sake of simplicity, until the end of this section we will adopt the notation $P_n^{[1]}(\cdot) := P_n^{[1]}(\cdot;\mathcal{I}_\xi)$ for $n\in\mathbb{N}$, unless the context requires more precision.

All the theory here presented is essentially based on the properties of the elements of the dual sequences associated to $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$. Therefore, we must know more about the transpose of $\mathcal{I}_\xi$, *i.e.* ${}^t\mathcal{I}_\xi$. Following (1.1.1)-(1.1.4), ${}^t\mathcal{I}_\xi := -x\,D + \xi\,\mathbb{I}_{\mathcal{P}'}$. The fact that either on $\mathcal{P}$ or in $\mathcal{P}'$ we have $Dx = xD + \mathbb{I}$, provides

$$\mathcal{I}_\xi = x\,D + (\xi+1)\mathbb{I}_{\mathcal{P}}$$

and therefore

$$ {}^{t}\mathcal{I}_{\xi} = -\mathcal{I}_{\xi} + (2\xi+1)\mathbb{I}_{\mathcal{P}'} \tag{4.1.4} $$

so, with a slight abuse of notation without consequence, $\mathcal{I}_{\xi}$ is defined either on $\mathcal{P}$ or on $\mathcal{P}'$.

**Lemma 4.1.1.** *Denoting by $\{u_n\}_{n\in\mathbb{N}}$ and $\{u_n^{[1]}\}_{n\in\mathbb{N}}$ the dual sequences associated to $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, respectively, we have the following relation:*

$$ -\,x\,\big(u_n^{[1]}\big)' + \xi\,u_n^{[1]} = (n+1+\xi)\,u_n\ , \quad n\in\mathbb{N}. \tag{4.1.5} $$

*Proof.* Since $\langle u_n^{[1]}, P_n^{[1]}\rangle = \delta_{n,m}$ and

$$ \begin{aligned} \langle u_n^{[1]}, P_m^{[1]}\rangle &= \frac{1}{m+1+\xi}\,\langle u_n^{[1]}\ ,\ \big(x\ P_m\big)' + \xi P_m\rangle \\ &= \frac{1}{m+1+\xi}\,\langle -x\,\big(u_n^{[1]}\big)' +\ \xi\,u_n^{[1]},\ P_m\rangle\ , \qquad n,m\geqslant 0, \end{aligned} $$

we must have

$$ \frac{1}{n+1+\xi}\,\langle -x\,\big(u_n^{[1]}\big)' +\ \xi\,u_n^{[1]},\ P_m\rangle = \delta_{n,m}\ , \qquad n,m\geqslant 0. $$

In accordance with lemma 1.3.1, we get: $\dfrac{1}{n+1+\xi}\Big(-\,x\ \big(u_n^{[1]}\big)' +\ \xi\ u_n^{[1]}\Big) = \sum_{\nu=0}^{n}\lambda_{n,\nu}\ u_\nu$ where $\lambda_{n,\nu} = \frac{1}{m+1+\xi}\langle -x\,\big(u_n^{[1]}\big)' +\ \xi\,u_n^{[1]},\ P_\nu\rangle$ for $0\leqslant\nu\leqslant n$, whence the result. □

The relation (4.1.5) may be also expressed like

$$ -\mathcal{I}_{\xi}(u_n^{[1]}) + (2\xi+1)u_n^{[1]} = (n+1+\xi)\,u_n\ , \quad n\in\mathbb{N}, $$

**Remark 4.1.1** (about the $\mathcal{I}_{\xi}$-Appell sequences)**.** Assume that $\{P_n\}_{n\in\mathbb{N}}$ possesses the $\mathcal{I}_{\xi}$-Appell character, meaning that $P_n = P_n^{[1]}$, for all the integers $n\in\mathbb{N}$. In this case, (4.1.1) becomes

$$ x\ P_n'(x) = n\ P_n, \quad n\in\mathbb{N}. $$

Consequently, the MPS $\{P_n\}_{n\in\mathbb{N}}$ is essentially (*i.e.* up to a linear change of variable) the sequence of monomials $\{x^n\}_{n\in\mathbb{N}}$, up to a shift. The impossibility of such sequence to be orthogonal shows the unfeasibility of $\mathcal{I}_{\xi}$-Appell orthogonal sequences.

### 4.1.1 Characterisation of classical sequences with relation to $\mathcal{I}_\xi$

The main goal is to find all the MOPS $\{P_n\}_{n\in\mathbb{N}}$ satisfying Hahn's property, or in other words, to search all the MOPS $\{P_n\}_{n\in\mathbb{N}}$ such that the MPS $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ defined by (4.1.1) is also orthogonal.

**Theorem 4.1.2.** *Consider The $\{P_n\}_{n\in\mathbb{N}}$ to be a MOPS with respect to the regular form $u_0$. The following statements are equivalent:*

*(a)* $\{P_n\}_{n\in\mathbb{N}}$ *is $\mathcal{I}_\xi$-classical .*

*(b)* *The elements of $\{P_n\}_{n\in\mathbb{N}}$ fulfil*

$$\mathcal{I}_\xi^* \, \mathcal{I}_\xi\Big(P_n(x)\Big) = (n+1+\xi)\lambda_n P_n(x), \quad n \in \mathbb{N}, \tag{4.1.6}$$

*with*

$$\mathcal{I}_\xi^* = K \; \Phi(x) D + \lambda_0 \mathbb{I} \tag{4.1.7}$$

*where $K$ and $\lambda_0$ represent two nonzero constants and $\Phi$ a monic polynomial satisfying* $\deg\Phi \leqslant 1$.

*(c)* *There exist a monic polynomial $\Phi(\cdot)$ and a nonzero constant $\lambda_0$ such that*

$$D\Big(x\Phi(x)u_0\Big) + \Psi(x)u_0 = 0 \tag{4.1.8}$$

*with*

$$\Psi(x) = -\big\{(2+\xi)\Phi(x) + \lambda_0 \; K^{-1} \; x\big\}, \tag{4.1.9}$$

$$\deg\Phi \leqslant 1 \; ; \; \deg\Psi = 1 \; ; \; \Phi(0) \; \Big(\Psi'(0) - (n-2-\xi)\Phi'(0)\Big) \neq 0, \; n \in \mathbb{N}. \tag{4.1.10}$$

*(d)* *There exist a monic polynomial $\Phi(\cdot)$ and a nonzero constant $\lambda_0$ such that*

$$\mathcal{I}_\xi\Big(\Phi(x) \; u_0\Big) + \Big\{\Psi(x) - \xi \; \Phi(x)\Big\} \; u_0 = 0 \; , \tag{4.1.11}$$

*and the conditions* (4.1.9)-(4.1.10) *are satisfied.*

*Proof.* The proof will be performed by showing that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a). The assumption of the MOPS $\{P_n\}_{n\in\mathbb{N}}$ being $\mathcal{I}_\xi$-classical supplies, according to its definition, the orthogonality of $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$. Therefore their elements ought to satisfy a second order recurrence relation, whose recurrence coefficients will be here denoted as $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$

and $(\beta_n^{[1]}, \gamma_{n+1}^{[1]})_{n\in\mathbb{N}}$, respectively. The elements of the corresponding dual sequences may be expressed by means of the first one, as follows:

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n\, u_0\ , \quad n\in\mathbb{N},$$
$$u_n^{[1]} = \left(\langle\, u_0^{[1]}\ , (P_n^{[1]})^2\, \rangle\right)^{-1} P_n^{[1]}\, u_0^{[1]}\ \ , \quad n\in\mathbb{N}.$$

Inserting these two last relations in (4.1.5) leads to

$$-\,x\,\left(P_n^{[1]}(x)\, u_0^{[1]}\ \right)' + \xi\, P_n^{[1]}(x)\, u_0^{[1]}\ = \lambda_n\, P_n(x)\, u_0\ , \quad n\in\mathbb{N}, \tag{4.1.12}$$

where

$$\lambda_n = (n+1+\xi)\frac{\langle u_0^{[1]}\ , {P_n^{[1]}}^2\rangle}{\langle u_0, P_n^2\rangle}\ , \quad n\in\mathbb{N}. \tag{4.1.13}$$

Naturally, based on the preliminary properties given in page 22, it is possible to transform (4.1.12) into

$$P_n^{[1]}(x)\left\{-\,x\,\left(u_0^{[1]}\right)' + \xi\, u_0^{[1]}\ \right\} - x\,\left(P_n^{[1]}(x)\right)' u_0^{[1]}\ = \lambda_n\, P_n(x)\, u_0\ , \quad n\in\mathbb{N}. \tag{4.1.14}$$

In particular, when $n=0$ from the previous identity we obtain

$$-\,x\,\left(u_0^{[1]}\right)' + \xi\, u_0^{[1]}\ = \lambda_0\, u_0\ . \tag{4.1.15}$$

providing (4.1.14) to become

$$-\,x\,\left(P_n^{[1]}(x)\right)'\, u_0^{[1]}\ = \left(\lambda_n\, P_n(x) - \lambda_0 P_n^{[1]}(x)\right) u_0\ , \quad n\in\mathbb{N}. \tag{4.1.16}$$

With the substitution of $n=1$ in this latter, we obtain

$$-\,x\, u_0^{[1]}\ = K\,\Phi(x)\, u_0 \tag{4.1.17}$$

where $K$ represents a nonzero constant such that the polynomial $\Phi$ defined through

$$K\,\Phi(x) = \lambda_1\, P_1(x) - \lambda_0 P_1^{[1]}(x) \tag{4.1.18}$$

is monic, and the subsequently replacement of the term $(-x\, u_0^{[1]}\ )$ in (4.1.16) yields

$$\left\{K\,\Phi(x)\left(P_n^{[1]}(x)\right)' - \lambda_n P_n(x) + \lambda_0 P_n^{[1]}(x)\right\} u_0 = 0\ , \quad n\in\mathbb{N}.$$

This together with the regularity of $u_0$ enables

$$K\,\Phi(x)\left(P_n^{[1]}(x)\right)' - \lambda_n P_n(x) + \lambda_0 P_n^{[1]}(x) = 0\ , \quad n\in\mathbb{N},$$

which, may be rewritten as

$$\mathcal{I}_\xi^* \left( P_n^{[1]}(x) \right) = \lambda_n P_n(x) \ , \quad n \in \mathbb{N}, \tag{4.1.19}$$

if we consider $\mathcal{I}_\xi^*$ to be the operator defined in (4.1.7). By definition, $P_n^{[1]}(x) := \frac{1}{n+1} \mathcal{I}_\xi \left( P_n(x) \right)$, thereby (4.1.19) provides (4.1.6). It shall be noticed that (4.1.6) also implies (4.1.19) for the same reason of the converse.

Let us now show that (b) implies (c). Equating the coefficients of the highest powers in $x$ on (4.1.19), we figure out the condition

$$K \, \Phi'(0) \, n + \lambda_0 = \lambda_n \neq 0, \quad n \in \mathbb{N},$$

because $\lambda_n \neq 0, n \in \mathbb{N}$. The action of $u_0$ over both sides of (4.1.6) corresponds to:

$$\big\langle u_0 \ , \ \mathcal{I}_\xi^* \, \mathcal{I}_\xi \ P_n(x) \big\rangle = \langle u_0 \ , \ (n+1+\xi)\lambda_n \ P_n(x) \rangle \ , \quad n \in \mathbb{N}.$$

which may be written as

$$\Big\langle \, u_0 \ , \ \mathcal{I}_\xi^* \, \mathcal{I}_\xi \ P_n(x) \, \Big\rangle = (1+\xi) \ \lambda_0 \ \delta_{n,0} \ , \quad n \in \mathbb{N}. \tag{4.1.20}$$

because $u_0$ is the first element of the dual sequence associated to $\{P_n\}_{n\in\mathbb{N}}$. By duality, we consider the transpose of the operator $\big( \mathcal{I}_\xi^* \, \mathcal{I}_\xi \big)$:

$$\begin{aligned} {}^t\big(\mathcal{I}_\xi^* \, \mathcal{I}_\xi\big) &= {}^t\mathcal{I}_\xi \, {}^t\mathcal{I}_\xi^* = \Big( -xD + \xi \mathbb{I} \Big) \Big( -KD\Phi(x) + \lambda_0 \mathbb{I} \Big) \\ &= \Big( -Dx + (1+\xi) \, \mathbb{I} \Big) \Big( -K \, D\Phi(x) + \lambda_0 \mathbb{I} \Big) \\ &= D \, \Big( K \, x \, D\Phi(x) - \big[ \lambda_0 \, x + K(1+\xi)\Phi(x) \big] \, \mathbb{I} \Big) + \lambda_0 (1+\xi) \, \mathbb{I} \end{aligned}$$

Consequently, (4.1.20) may be transformed into

$$\bigg\langle \Big( K \, x \, (\Phi(x) u_0)' - \big[ \lambda_0 \, x + K(1+\xi)\Phi(x) \big] u_0 \Big)' + \lambda_0 (\xi+1) u_0, P_n(x) \bigg\rangle = (1+\xi) \ \lambda_0 \ \delta_{n,0},$$

with $n \in \mathbb{N}$. This corresponds to

$$\bigg\langle \Big( K \, x \, (\Phi(x) u_0)' - \big[ \lambda_0 \, x + K(1+\xi)\Phi(x) \big] u_0 \Big)', P_n(x) \bigg\rangle = 0 \ , \quad n \in \mathbb{N},$$

which, because $\{P_n\}_{n\in\mathbb{N}}$ spans $\mathcal{P}$, compels to have

$$\Big( K \, x \, (\Phi(x) u_0)' - \big[ \lambda_0 \, x + K(1+\xi)\Phi(x) \big] u_0 \Big)' = 0 \ .$$

The injectivity of the derivative operator over $\mathcal{P}'$ allows the conclusion

$$K \, x \, (\Phi(x) u_0)' - \big[ \lambda_0 \, x + K(1+\xi)\Phi(x) \big] u_0 = 0 \ ,$$

and, by setting $\Psi(\cdot)$ as in (4.1.9), this latter equation may be rewritten as in (4.1.8). The remaining conditions of (4.1.10) are indeed a natural consequence of (4.1.8), regarding that it is equivalent to the recurrence relation that furnishes the moments of $u_0$:

$$\Big(D\big(x\ \Phi(x)\ u_0\big)\Big)_n - \Big(\big((2+\xi)\Phi(x) + \lambda_0\ K^{-1}\ x\big)u_0\Big)_n = 0\ , \quad n \in \mathbb{N}.$$

that is,

$$\big\langle\ D\big(x\ \Phi(x)\ u_0\big) - \big((2+\xi)\Phi(x) + \lambda_0\ K^{-1}\ x\big)u_0\ ,\ x^n\ \big\rangle = 0\ , \quad n \in \mathbb{N},$$

which corresponds to

$$\big\langle\ u_0\ ,\ -(n+2+\xi)x^n\Phi(x) - \lambda_0\ K^{-1}x^{n+1}\ \big\rangle = 0\ , \quad n \in \mathbb{N},$$

As it is always possible to write $\Phi(x) = \Phi'(0)x + \Phi(0)$, this latter brings

$$-\Big\{(n+2+\xi)\Phi'(0) + \lambda_0 K^{-1}\Big\}(u_0)_{n+1} - (n+2+\xi)\ \Phi(0)(u_0)_n = 0, \quad n \in \mathbb{N},$$

or, based on the definition of $\Psi$ given by (4.1.9), we have

$$-\Big\{n\,\Phi'(0) - \Psi'(0)\Big\}(u_0)_{n+1} - (n+2+\xi)\ \Phi(0)(u_0)_n = 0, \quad n \in \mathbb{N}.$$

Based on the regularity of $u_0$, we necessarily have $\Phi(0) \neq 0$ and therefore $n\,\Phi'(0) - \Psi'(0) \neq 0$ for any nonnegative integer $n$. In particular, it follows that $\Psi'(0) \neq 0$, ergo $\deg \Psi = 1$.

According to the definition of the operator $\mathcal{I}_\xi$, the equation in $u_0$ (4.1.8) admits the claimed representation given by (4.1.11), whence we have just proved (c) $\Leftrightarrow$ (d).

At last, we shall show that (d) $\Rightarrow$ (a). Let us suppose that the regular form $u_0$ associated to $\{P_n\}_{n\in\mathbb{N}}$ fulfils (4.1.11) with $\deg \Phi \leqslant 1$ and $\Psi(\cdot)$ given by (4.1.9), that is, $u_0$ fulfils

$$\mathcal{I}_\xi\big(K\Phi(x)u_0\big) - \Big\{2(1+\xi)K\Phi(x) + \lambda_0 x\Big\}u_0 = 0\ . \tag{4.1.21}$$

Upon this, we aim to show the MPS $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ to be orthogonal with respect to the form $v$ given by $(2+\xi)v = -\,(K\,\Phi(x)\,u_0)' + \lambda_0\,u_0$. Hence, we successively have for any integers $n, m$ such that $0 \leqslant m \leqslant n$:

$$\begin{aligned}\Big\langle\ v\ ,\ x^m\ P_n^{[1]}(x)\ \Big\rangle &= \frac{1}{n+1+\xi}\,\langle\ v\ ,\ x^m\ \ (\mathcal{I}_\xi P_n)(x)\ \rangle \\ &= \frac{1}{n+1+\xi}\,\big\langle\ v\ ,\ x^m\{(x\,P_n(x))' + \xi\,P_n(x)\}\ \big\rangle \\ &= \frac{1}{n+1+\xi}\,\Big\langle\ v\ ,\ \big(x^{m+1}\,P_n(x)\big)' - (m-\xi)x^m\,P_n(x)\ \Big\rangle \\ &= \frac{1}{n+1+\xi}\,\langle\ -x\,v' - (m-\xi)v\ ,\ x^m\,P_n(x)\ \rangle\end{aligned} \tag{4.1.22}$$

From the definition of the form $v$, we successively deduce

$$\begin{aligned}
-x\,v' - (m-\xi)v &= \frac{1}{(1+\xi)}\Big\{ -x\Big( -(K\,\Phi(x)\,u_0)' + \lambda_0\,u_0\Big)' \\
&\quad -(m-\xi)\Big( -(K\,\Phi(x)\,u_0)' + \lambda_0\,u_0\Big)\Big\} \\
&= \frac{1}{(1+\xi)}\Big\{\Big[\big(K\,x\,\Phi(x)u_0\big)' - \{(2+\xi)K\Phi(x) + \lambda_0 x\}u_0\Big]' \\
&\quad +m\big(K\Phi(x)u_0\big)' - (m-\xi-1)\lambda_0 u_0\Big\} \\
&= \frac{1}{(1+\xi)}\Big\{\Big[\mathcal{I}_\xi\big(K\,x\,\Phi(x)u_0\big) - \{2(1+\xi)K\Phi(x) + \lambda_0 x\}u_0\Big]' \\
&\quad +m\big(K\Phi(x)u_0\big)' - (m-\xi-1)\lambda_0 u_0\Big\}
\end{aligned}$$

and after (4.1.21), we obtain

$$-x\,v' - (m-\xi)v = \frac{1}{(1+\xi)} m\big(K\Phi(x)u_0\big)' - (m-\xi-1)\lambda_0 u_0\ , \quad 0\leqslant m\leqslant n\,.$$

Consequently, equating the first and last members of (4.1.22), we obtain

$$\Big\langle v, x^m\,P_n^{[1]}(x)\Big\rangle = \frac{1}{(n+1+\xi)(1+\xi)}\Big\langle m\,x\,\big(K\Phi(x)u_0\big)' - (m-\xi-1)\lambda_0\,x\,u_0, x^{m-1}\,P_n(x)\Big\rangle$$

for any $m,n\in\mathbb{N}$ with $0\leqslant m\leqslant n$. The case where $m=0$ brings

$$\Big\langle v, P_n^{[1]}(x)\Big\rangle = \frac{\lambda_0}{(1+\xi)}\langle\, u_0\ ,\ P_n(x)\,\rangle = \frac{\lambda_0}{(1+\xi)}\,\delta_{n,0}\ ,\ n\in\mathbb{N},$$

while the case of $m\geqslant 1$ with $m\leqslant n$ leads to

$$\begin{aligned}
\Big\langle v, x^m\,P_n^{[1]}(x)\Big\rangle =& \tfrac{1}{(n+1+\xi)(1+\xi)}\Big\{\Big\langle m\big[\mathcal{I}_\xi\big(K\Phi(x)u_0\big) - K(1+\xi)\Phi(x)u_0\big], x^{m-1}\,P_n(x)\Big\rangle \\
&+(1+\xi-m)\lambda_0\Big\langle u_0\ ,\ x^m\,P_n(x)\ \Big\rangle\Big\}\ ,\ 1\leqslant m\leqslant n,
\end{aligned}$$

which, according to (4.1.21), corresponds to

$$\Big\langle v, x^m\,P_n^{[1]}(x)\ \Big\rangle = \tfrac{1}{(n+1+\xi)(1+\xi)}\Big\langle u_0, \big[m\,K\,(1+\xi)\Phi(x)u_0 + \lambda_0(1+\xi)\,x\big]x^{m-1}\,P_n(x)\Big\rangle\ .$$

As a result, we have

$$\Big\langle\ v\ ,\ x^m\,P_n^{[1]}(x)\ \Big\rangle = \frac{1}{(n+1+\xi)}\Big\langle u_0, \big[m\,K\,\Phi(x)u_0 + \lambda_0\,x\big]x^{m-1}\,P_n(x)\ \Big\rangle\ , \quad 0\leqslant m\leqslant n\ .$$

Inasmuch as $\{P_n\}_{n\in\mathbb{N}}$ is a MOPS, proposition 1.4.2 permits to deduce from the previous

$$\Big\langle\ v\ ,\ x^m\,P_n^{[1]}(x)\ \Big\rangle = \frac{1}{(n+1+\xi)}\Big(n\,K\,\Phi'(0) + \lambda_0\Big)\delta_{n,m}\ , \quad 0\leqslant m\leqslant n\ .$$

Under the assumptions, we have $n\,K\,\Phi'(0)+\lambda_0\neq 0$ for all $n\in\mathbb{N}$. Again, based on proposition 1.4.2, the orthogonality of $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is assured. □

**Remark 4.1.2.** As noticed in theorem 4.1.2, the elements of a $\mathcal{I}_\xi$-classical sequence fulfil (4.1.6). In this case the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is orthogonal and based on its definition given by (4.1.1), this relation corresponds to (4.1.19). Operating with $\mathcal{I}_\xi$ over both sides of this latter leads to

$$\mathcal{I}_\xi\, \mathcal{I}_\xi^*\left(P_n^{[1]}(x)\right) = (n+1+\xi)\lambda_n P_n^{[1]}(x), \quad n\in\mathbb{N}. \tag{4.1.23}$$

As a matter of fact, under (4.1.1), the relations (4.1.6) and (4.1.23) are equivalent. Moreover, the first one corresponds to

$$x\,\Phi(x)\,P_n''(x) - \Psi(x)P_n'(x) = K^{-1}\Big\{(n+1+\xi)\lambda_n - (1+\xi)\lambda_0\Big\}P_n(x)\,, \quad n\in\mathbb{N},$$

whereas the second one corresponds to

$$\begin{aligned} & x\,\Phi(x)\left(P_n^{[1]}(x)\right)'' - \Big(\Psi(x) - (x\Phi(x))'\Big)\left(P_n^{[1]}(x)\right)' \\ & = K^{-1}\big\{(n+1+\xi)\lambda_n - (1+\xi)\lambda_0\big\}\; P_n^{[1]}(x)\,, \quad n\in\mathbb{N}. \end{aligned}$$

As a consequence of theorem 4.1.2, the equivalence between statements (a) and (c) compels us to conclude that the $\mathcal{I}_\xi$-classical forms must be either Laguerre or Jacobi forms, depending on whether $\deg\Phi = 0$ or $\deg\Phi = 1$. This brings an alternative characterisation of these two classical sequences.

#### 4.1.1.1 About the invariance of the $\mathcal{I}_\xi$-classical character by a linear transformation

As previously pointed out on the example of page 30, the MPS $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$ defined by

$$\widetilde{P}_n(x) = a^{-n}P_n(ax+b), \text{ with } a\in\mathbb{C}^*, b\in\mathbb{C},\, n\in\mathbb{N}.$$

is orthogonal with respect to the form $\widetilde{u}_0 = \big(h_{a^{-1}}\circ\tau_{-b}\big)u_0$ as long as $\{P_n\}_{n\in\mathbb{N}}$ is a MOPS with respect to $u_0$.

Insofar as the MOPS $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}_\xi$-classical, the MOPS $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$ also is, since the regular form $\widetilde{u}_0$ satisfies

$$D\Big(a^{-1}(ax+b)\widetilde{\Phi}(x)\widetilde{u}_0\Big) + \widetilde{\Psi}(x)\widetilde{u}_0 = 0$$

with

$$\widetilde{\Phi}(x) = a^{-\deg\Phi}\Phi(ax+b) \quad ; \quad \widetilde{\Psi}(x) = a^{-\deg\Phi}\Psi(ax+b)\,,$$

and therefore, theorem 4.1.2 guarantees the $\mathcal{I}_\xi$-classical character the corresponding orthogonal sequence $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$.

The reason behind this lies essentially on the fact that any affine transformation leaves invariant the (semi)-classical character (please consult page 70).

#### 4.1.1.2 About the sequence of the $\mathcal{I}_\xi$-derivatives

It remains to know whether the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is also $\mathcal{I}_\xi$-classical whenever the sequence $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}_\xi$-classical. Therefore we shall make some additional analysis over the form $u_0^{[1]}$.

**Lemma 4.1.3.** *If $u_0$ is a $\mathcal{I}_\xi$-classical form, then there exists a monic polynomial $\Phi(\cdot)$ and a polynomial $\Psi(\cdot)$ such that the regular form $u_0^{[1]}$*

$$D\Big(x\ \Phi(x)\ u_0^{[1]}\Big) + \Big\{\Psi(x) - x\ \Phi'(x) + \Phi(x)\Big\}u_0^{[1]} = 0 \tag{4.1.24}$$

*and the pair $\Big(\Phi(x)\ ,\ \Psi(x) - x\ \Phi'(x) + \Phi(x)\Big)$ satisfies the conditions* (4.1.10).

*Proof.* Following the proof of theorem 4.1.2, the $\mathcal{I}_\xi$-classical character of the regular form $u_0$ provides the regular form $u_0^{[1]}$ to be related with $u_0$ through the conditions (4.1.15) and (4.1.17). Thus, between these two conditions it is possible to eliminate the term in $u_0$. This procedure leads to

$$K\ \Phi(x)\left(x\left(u_0^{[1]}\right)' - \xi\ u_0^{[1]}\right) - \lambda_0\ x\ u_0^{[1]}\ = 0$$

with the nonzero constant $K$ and the polynomial $\Phi$ defined according to (4.1.18). The precedent equation in $u_0^{[1]}$ may be equivalently written like

$$\Big(x\ \Phi(x)\ u_0^{[1]}\Big)' - \big\{\lambda_0\ K^{-1}\ x + x\Phi'(x) + (1+\xi)\Phi(x)\big\}\ u_0^{[1]} = 0$$

which, after setting $\Psi(\cdot)$ to be the polynomial defined in (4.1.9), corresponds to (4.1.24). □

The previous lemma together with Proposition 4.1.2 brings to light that the MOPS $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is $\mathcal{I}_\xi$-classical, as long as $\{P_n\}_{n\in\mathbb{N}}$ is. More generally, for some $k\in\mathbb{N}^*$ consider the sequence $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ recursively defined by $P_n^{[k+1]}(x) = (n+1+\xi)\left(\mathcal{I}_\xi P_n^{[k]}\right)(x)$, $n\in\mathbb{N}$, with the convention $P_n^{[0]}(\cdot) := P_n(\cdot)$. By finite induction and according to previous lemma, we conclude that if $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}_\xi$-classical, then the sequence $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is orthogonal and its corresponding regular form $u_0^{[k]}$ fulfils

$$D\Big(x\ \Phi(x)\ u_0^{[k]}\Big) + \Big\{\Psi(x) - k\,x\ \Phi'(x) + k\,\Phi(x)\Big\}\ u_0^{[k]} = 0\ ,$$

According to Proposition 4.1.2, the sequence $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is also $\mathcal{I}_\xi$-classical. Conversely, when the MOPS $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is $\mathcal{I}_\xi$-classical, the same occurs with $\{P_n\}_{n\in\mathbb{N}}$.

In this case, denoting by $u_0^{[k]}$ the regular form associated to the $\mathcal{I}_\xi$-classical sequence $\{P_n^{[k]}\}_{n\in\mathbb{N}}$, both $u_0$ and $u_0^{[k]}$ are either a Laguerre or Jacobi form. In order to have a more precise information about the range for the parameters of the Laguerre or Jacobi forms, we now turn our analysis towards the search of the possible expressions for the first recurrence coefficients associated to the two MOPS $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ which will enable to compute the polynomials $\Phi$ and $\Psi$ presented in (4.1.8).

### 4.1.2 Construction of the $\mathcal{I}_\xi$-classical polynomial sequences

Suppose $\{P_n\}_{n\in\mathbb{N}}$ is a $\mathcal{I}_\xi$-classical polynomial sequence. As both $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ are orthogonal, they fulfil the second order recurrence relations

$$\begin{cases} P_0(x) = 1 \quad ; \quad P_1(x) = x - \beta_0 \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x)\ , \quad n \in \mathbb{N}, \end{cases}$$

and

$$\begin{cases} P_0^{[1]}(x) = 1 \quad ; \quad P_1^{[1]}(x) = x - \beta_0 \\ P_{n+2}^{[1]}(x) = (x - \beta_{n+1}^{[1]})P_{n+1}^{[1]}(x) - \gamma_{n+1}^{[1]}P_n^{[1]}(x)\ , \quad n \in \mathbb{N}, \end{cases}$$

with $\gamma_{n+1}\,\gamma_{n+1}^{[1]} \neq 0$, $n \in \mathbb{N}$.

We could now proceed to the determination of a system of equations fulfilled by the recurrence coefficients $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$, through an analogous approach of the one taken by Maroni in [78] while the author characterised the classical sequences. Nonetheless, since we already know that the $\mathcal{I}_\xi$-classical forms are classical, we only need to determine the first recurrence coefficients, but first, let us notice that from (4.1.13) we have

$$\lambda_0 = 1 + \xi$$
$$\lambda_{n+1} = \frac{(n+2+\xi)}{(n+1+\xi)}\,\frac{\gamma_{n+1}^{[1]}}{\gamma_{n+1}}\,\lambda_n\ , \quad n \in \mathbb{N}.$$

To solve this issue, we may use the relation (4.1.15), providing the moment equality

$$(n+1+\xi)(u_0^{[1]}\,)_n = \lambda_0 (u_0)_n\ , \quad n \in \mathbb{N}. \tag{4.1.25}$$

The case where $n = 0$ produces the known relation $\lambda_0 = \xi + 1$, whereas the particular choice of $n = 1$ leads to

$$\beta_0^{[1]} = \frac{\xi+1}{2+\xi}\,\beta_0\ .$$

Now, again from (4.1.25) with $n = 2$, we deduce $\gamma_1^{[1]} = \dfrac{3+\xi}{2+\xi}\left\{\dfrac{1}{(2+\xi)^2}{\beta_0}^2 + \gamma_1\right\}$ . Since, according to its definition, $\lambda_1 = (2+\xi)\,\frac{\gamma_1^{[1]}}{\gamma_1}$, we thus have

$$\lambda_1 = \frac{(\xi+1)}{(3+\xi)(2+\xi)}\,\left\{\frac{{\beta_0}^2}{\gamma_1} + (2+\xi)^2\right\} \tag{4.1.26}$$

and also

$$\lambda_1 - \lambda_0 = \frac{(\xi+1)}{(3+\xi)(2+\xi)} \left\{ \frac{{\beta_0}^2}{\gamma_1} - (2+\xi) \right\} \tag{4.1.27}$$

Hence, following (4.1.18), we have

$$K\ \Phi(x) = \frac{(\xi+1)}{(3+\xi)(2+\xi)} \left\{ \frac{{\beta_0}^2}{\gamma_1} - (2+\xi) \right\} x - \frac{(\xi+1)}{(3+\xi)} \beta_0 \left( \frac{{\beta_0}^2}{\gamma_1} + 1 \right)$$

and the polynomial $\Psi(\cdot)$ defined according to (4.1.9) becomes

$$K\ \Psi(x) = -\frac{1+\xi}{3+\xi}\,(\beta_0^2 + \gamma_1)\Big(x - \beta_0\Big)$$

where $K$ represents the nonnegative constant such that $\Phi(\cdot)$ is monic.

**Case I.** $\deg \Phi = 0$

Under this condition, we have $\gamma_1 = \dfrac{\beta_0^2}{2+\xi}$, therefore $\Phi(x) = 1$, $K = -\frac{1+\xi}{2+\xi}\ \beta_0 \neq 0$ and the polynomial $\Psi$ is given by

$$\Psi(x) = \frac{2+\xi}{\beta_0}\ x - (2+\xi) \tag{4.1.28}$$

As a result, the form $\widetilde{u}_0 = h_{a^{-1}}\, u_0$ with $a = \frac{\beta_0}{2+\xi}$ fulfils[1]

$$D\big(x\, \widetilde{u}_0\big) + \Big(x - (2+\xi)\Big)\widetilde{u}_0 = 0\,.$$

According to the information provided in Table 2.1, we conclude that $\widetilde{u}_0$, just like $u_0$, is a Laguerre form of parameter $(\xi+1)$, and the associated MOPS $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$ with $\widetilde{P}_n(x) := a^{-n} P_n(a\,x)$ (for $n \in \mathbb{N}$) is a Laguerre polynomial sequence of parameter $(\xi+1)$. The well known recurrence coefficients, say $(\widetilde{\beta}_n, \widetilde{\gamma}_{n+1})_{n\in\mathbb{N}}$, associated to $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$ are listed in Table 2.1, whence we deduce:

$$\beta_n = a\,\widetilde{\beta}_n = \frac{\beta_0}{2+\xi}\,(2n+\xi+2) \quad ; \quad \gamma_{n+1} = a^2\,\widetilde{\gamma}_{n+1} = \frac{{\beta_0}^2}{(2+\xi)^2}\,(n+1)(n+\xi+2)\,, \quad n \in \mathbb{N}.$$

On the other hand, following (4.1.24) and after the precedent conclusions, the form $u_0^{[1]}$ fulfils

$$D\left(x\, u_0^{[1]}\right) + \Big(\frac{2+\xi}{\beta_0}\, x - (1+\xi)\Big)\, u_0^{[1]} = 0\,,$$

so, setting $a = \frac{\beta_0}{2+\xi}$, the form $\widetilde{u}_0^{[1]} = h_{a^{-1}}\, u_0^{[1]}$ fulfils

$$D\left(x\, \widetilde{u}_0^{[1]}\right) + \Big(x - (1+\xi)\Big)\, \widetilde{u}_0^{[1]} = 0\,.$$

[1]For more details about the invariance of the classical character under an affine transformation please consult p.34.

Once again the information given in Table 2.1 allows us to deduce that $\widetilde{u}_0^{[1]}$, just like $u_0^{[1]}$, is a Laguerre form with parameter $\xi$, and, of course the associated MOPS $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is a Laguerre polynomial sequence with parameter $\xi$.

**Case II.** $\deg\Phi = 1$

In this case we have $\Phi(x) = (x-r)$ , with

$$r = \frac{\beta_0\,(1+\xi)}{(2+\xi)(3+\xi)\,K}\;\left(\frac{\beta_0^2}{\gamma_1}+1\right)\;,$$

The nonzero constant $k$ is thus given by

$$K = \frac{(\xi+1)}{(3+\xi)(2+\xi)}\;\left\{\frac{{\beta_0}^2}{\gamma_1}-(2+\xi)\right\}$$

and

$$\Psi(x) = -\left(\frac{(\xi+1)}{k}+(2+\xi)\right)\;x+r(2+\xi)\;.$$

In this case, $u_0$ is a classical form of Jacobi and so is the form $\widetilde{u}_0 = (h_{a^{-1}}\circ\tau_{-a})\;u_0$ with $a=\frac{r}{2}$, because of the invariance of the classical character under an affine transformation (please consult p.34), whence we deduce that the form $\widetilde{u}_0$ satisfies the equation

$$D\Big((x^2-1)\widetilde{u}_0\Big)+\Big(-\frac{(\xi+1+K(2+\xi))}{k}\;x-\frac{\xi+1-K(2+\xi)}{k}\Big)\,\widetilde{u}_0=0$$

Therefore introducing two new variables $\alpha,\beta$ and setting

$$\xi=\alpha-1 \qquad \text{and} \qquad K=\frac{\alpha}{\beta+1}$$

or, equivalently,

$$\alpha=\xi+1 \qquad \text{and} \qquad \beta=\frac{\xi+1-K}{K}$$

we conclude that $\widetilde{u}_0$, as well as $u_0$ (see p. 34), is a Jacobi form of parameters $(\alpha,\beta)$ but with the restriction over the range of orthogonality for a Jacobi form of $\alpha\neq 0$.

Besides, from lemma 4.1.3, $u_0^{[1]}$ fulfils (4.1.24), which may be expressed as

$$D\Big(x(x-r)u_0^{[1]}\Big)+\Big(-(2+\alpha+\beta)\,x+r\,\alpha\Big)\,u_0^{[1]}\;=0\;.$$

Consequently, the form $\widetilde{u}_0^{[1]} = (h_{a^{-1}}\circ\tau_{-a})\;u_0^{[1]}$ , with $a=\frac{r}{2}$, fulfils

$$D\Big((x^2-1)\widetilde{u}_0^{[1]}\Big)+\Big(-(2+\alpha+\beta)\,x+\alpha-\beta-2\Big)\,\widetilde{u}_0^{[1]}=0\;,$$

allowing to conclude that both $u_0^{[1]}$ and $\widetilde{u}_0^{[1]}$ are Jacobi forms of parameters $(\alpha-1,\beta+1)$ while $u_0$ is a Jacobi form of parameters $(\alpha,\beta)$ with $\alpha\neq 0$.

### 4.1.3 Some comments on the $\mathcal{I}_\xi$-classical sequences

So far we achieved the following conclusions:

- If $\{P_n\}_{n\in\mathbb{N}}$ is a $\mathcal{I}_\xi$-classical sequence, then it is either a Laguerre sequence of parameter $(\xi+1)$ or a Jacobi sequence of parameters $(\xi+1, \frac{\xi+1}{\mu}-1)$ with $\mu \neq 0$ and $\xi \neq -(n+1)$, for $n \in \mathbb{N}$. For instance, the Laguerre sequences of parameter $0$ or the Legendre sequences cannot be $\mathcal{I}_\xi$-classical sequences.
- Whenever $\{P_n\}_{n\in\mathbb{N}}$ is a $\mathcal{I}_\xi$-classical sequence, then so is $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$. In the case where $\{P_n\}_{n\in\mathbb{N}}$ is a Laguerre sequence of parameter $(\xi+1)$, $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$ is a Laguerre sequence of parameter $\xi$. On the other hand, as long as $\{P_n\}_{n\in\mathbb{N}}$ a Jacobi sequence of parameters $(\xi+1, \frac{\xi+1}{\mu}-1)$ (with $\mu\neq 0$ and $\xi \neq -(n+1)$, for $n\in\mathbb{N}$), $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$ is a Jacobi sequence of parameters $(\xi, \frac{\xi+1}{\mu})$.

Let us consider the monic sequence of derivatives of $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$, here denoted as $\{Q_n\}_{n\in\mathbb{N}}$ and defined through

$$Q_n(x) = \frac{1}{(n+1)} \left(P_{n+1}^{[1]}(x;\mathcal{I}_\xi)\right)' \ , \quad n\in\mathbb{N}.$$

Clearly, the relation between the elements of $\{Q_n\}_{n\in\mathbb{N}}$ and those of $\{P_n\}_{n\in\mathbb{N}}$ is given by

$$Q_n(x) = \frac{1}{(n+1)(n+2+\xi)} \ (\mathcal{I}_\xi P_{n+1}(x))' \ , \quad n\in\mathbb{N},$$

which may be equivalently expressed as follows

$$Q_n(x) = \frac{1}{(n+1)(n+2+\xi)} \left(\big[DxD + (\xi+1)D\big]P_{n+1}\right)(x) \ , \quad n\in\mathbb{N}.$$

Recalling (3.1.5) and (3.1.19), this last equality provides

$$Q_n(x) = P_n^{[1]}(x;\mathcal{F}_{2(\xi+1)}) \ , \quad n\in\mathbb{N}.$$

Now suppose the MOPS $\{P_n\}_{n\in\mathbb{N}}$ to be $\mathcal{I}_\xi$-classical, which corresponds to assume that both $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$ are orthogonal sequences. As we have seen, necessarily either $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$ is a $(D)$-classical sequence of Laguerre of parameter $\xi$ or it matches a $(D)$-classical sequence of Jacobi of parameters $(\xi, \frac{\xi+1}{\mu})$ with $\xi \neq -(n+1)$ and $\mu\neq 0$ which implies, according to the considerations made on section 2.1.2 (pp.37-37), the MPS $\{P_n^{[1]}(\cdot;\mathcal{F}_{2(\xi+1)})\}_{n\in\mathbb{N}}$ to be a Laguerre sequence of parameter $(\xi+1)$ or a Jacobi sequence of parameters $(\xi+1, \frac{\xi+1}{\mu}+1)$, respectively.

To sum up, the $\mathcal{I}_\xi$-classical character of a sequence $\{P_n\}_{n\in\mathbb{N}}$ implies its $\mathcal{F}_{2(\xi+1)}$-classical character, since the orthogonality of $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}(\cdot;\mathcal{I}_\xi)\}_{n\in\mathbb{N}}$ supplies the orthogonality of $\{P_n^{[1]}(\cdot;\mathcal{F}_{2(\xi+1)})\}_{n\in\mathbb{N}}$. Furthermore, we have achieved the conclusion that the Laguerre sequence of parameter $\varepsilon/2$ and the Jacobi sequence of parameters $(\varepsilon/2, \frac{\varepsilon}{2\mu}+1)$ ( $\varepsilon,\mu\in\mathbb{C}$ such that $\varepsilon\neq -2n$, $n\in\mathbb{N}$, and $\mu\neq 0$) are not only $\mathcal{I}_{(\frac{\varepsilon}{2}-1)}$-classical but also $\mathcal{F}_\varepsilon$-classical.

Of course, it is not possible to assert in general that the $\mathcal{F}_\varepsilon$-classical character implies the $\mathcal{I}_{(\frac{\varepsilon}{2}-1)}$-classical character.

The forthcoming developments are concerned with the characterisation of all the $\mathcal{F}_\varepsilon$-classical sequences, where $\varepsilon$ represents a complex parameter different from any negative even integer.

## 4.2 The second order (Laguerre) differential operator

Consider the operator $\mathcal{F}_\varepsilon$ already defined by (3.1.5) and let $\{P_n\}_{n\in\mathbb{N}}$ be a MPS. In accordance with (3.1.19) it is possible to construct another MPS $\{P_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ whose elements are such that

$$P_n^{[1]}(x;\mathcal{F}_\varepsilon) = \frac{1}{\rho_{n+1}(\varepsilon)}\,\mathcal{F}_\varepsilon\Big(B_{n+1}(x)\Big),\quad n\in\mathbb{N}, \tag{4.2.1}$$

with

$$\rho_{n+1} := \rho_{n+1}(\varepsilon) = (n+1)\big(2(n+1)+\varepsilon\big),\quad n\in\mathbb{N}, \tag{4.2.2}$$

where $\varepsilon$ represents a complex parameter such that

$$\varepsilon\neq -2(n+1)\,,\; n\in\mathbb{N},\;. \tag{4.2.3}$$

On section 3.2 of chapter 3, we have presented some properties of this lowering operator $\mathcal{F}_\varepsilon$. Moreover, lemma 3.2.1 provides a relation fulfilled by the dual sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{u_n^{[1]}(\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$, respectively associated to $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, which was given in (3.2.3).

Subsequently, in section 3.3 (*see p.*93) we dealt with the problem of finding all the orthogonal sequences $\{P_n\}_{n\in\mathbb{N}}$ such that $\{P_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ coincides with the first one. Actually, we were searching a particular collection of the $\mathcal{F}_\varepsilon$-classical sequences, the $\mathcal{F}_\varepsilon$-Appell ones. For the moment we intend to attain all the MOPS $\{P_n\}_{n\in\mathbb{N}}$ such that $\{P_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)\}_{n\in\mathbb{N}}$ is also orthogonal. In other words, following definition 4.0.5, we aim to find all the $\mathcal{F}_\varepsilon$-classical sequences.

> Please note that, for the sake of simplicity, until the end of this section we will adopt the notation $P_n^{[1]}(\cdot) := P_n^{[1]}(\cdot;\mathcal{F}_\varepsilon)$ for $n\in\mathbb{N}$, unless the context requires more precision.

### 4.2.1 Characterisation of the $\mathcal{F}_\varepsilon$-classical sequences

The combination of the orthogonal properties of the sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ with the relation (3.2.3) yields the following result, crucial for the forthcoming developments.

**Lemma 4.2.1.** *Let $\{P_n\}_{n\in\mathbb{N}}$ be a MOPS with respect to $u_0$. When $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, defined by (4.2.1), is also a MOPS (with respect to $u_0^{[1]}$), then it holds:*

$$\mathcal{F}_{-\varepsilon}\left(P_n^{[1]}u_0^{[1]}\right) = \lambda_n(\varepsilon)\, P_{n+1}\, u_0, \quad n\in\mathbb{N}, \tag{4.2.4}$$

*where*

$$\lambda_n := \lambda_n(\varepsilon) = \rho_{n+1}\,\frac{\left\langle u_0^{[1]}, \left(P_n^{[1]}\right)^2\right\rangle}{\left\langle u_0, P_{n+1}^2\right\rangle}, \quad n\in\mathbb{N}, \tag{4.2.5}$$

*where $\rho_{n+1}$, $n\in\mathbb{N}$, is given by (4.2.2).*

*Proof.* According to the properties of a MOPS (see p.28), the terms of the dual sequences of $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ may be respectively expressed as $u_n = \langle u_0, P_n^2\rangle^{-1}P_n u_0,\ n\in\mathbb{N}$, and $u_n^{[1]} = \langle u_0^{[1]}, \left(P_n^{[1]}\right)^2\rangle^{-1}P_n^{[1]}u_0^{[1]},\ n\in\mathbb{N}$. Now, (3.2.3) allows us to show that (4.2.4) is satisfied. □

Based on the relation (4.2.4) we will establish functional relations fulfilled by the two forms $u_0$ and $u_0^{[1]}$. This will allow to get functional equations fulfilled by the form $u_0$ assuring that such form is definitely a semi-classical form. As a consequence, we will be able to define the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ by means of $\{P_n\}_{n\in\mathbb{N}}$.

**Lemma 4.2.2.** *Let $\{P_n\}_{n\in\mathbb{N}}$ be a MOPS with respect to $u_0$. When $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, defined by (4.2.1), is also a MOPS (with respect to $u_0^{[1]}$), then it holds:*

$$\mathcal{F}_{-\varepsilon}u_0^{[1]} = \lambda_0 P_1\, u_0 \tag{4.2.6}$$
$$(2-\varepsilon)u_0^{[1]} + 4x\,(u_0^{[1]})' = f(x;\varepsilon)u_0 \tag{4.2.7}$$
$$4x\,u_0^{[1]} = h(x;\varepsilon)u_0 \tag{4.2.8}$$

*where*

$$f(x) := f(x;\varepsilon) := A_2(x;\varepsilon) \tag{4.2.9}$$
$$h(x) := h(x;\varepsilon) := A_3(x;\varepsilon) - \left(P_2^{[1]}\right)'(x)A_2(x;\varepsilon) \tag{4.2.10}$$
$$A_{n+1}(\cdot;\varepsilon) = \lambda_n(\varepsilon)P_{n+1}(\cdot) - \lambda_0(\varepsilon)P_1(\cdot)P_n^{[1]}(\cdot), \quad n\in\mathbb{N}. \tag{4.2.11}$$

*In addition, we have*

$$n\,(n-1)\,\frac{1}{6}\,\frac{d^3}{dx^3}h(0) + n\,\frac{d^2}{dx^2}f(0) + 2\lambda_0 \neq 0,\ n\in\mathbb{N}. \tag{4.2.12}$$

*Proof.* Suppose $\{P_n\}_{n\in\mathbb{N}}$ is a MOPS and $u_0$ its regular form. In accordance with lemma 4.2.1 we have the relation (4.2.4), which, due to (3.2.2), may be rewritten as

$$P_n^{[1]}(\mathcal{F}_{-\varepsilon}u_0^{[1]}) - \big(\mathcal{F}_\varepsilon P_n^{[1]}\big)u_0^{[1]} + 4\Big(x\left(P_n^{[1]}\right)'(u_0^{[1]})\Big)' = \lambda_n(\varepsilon)P_{n+1}u_0, \quad n\in\mathbb{N}. \tag{4.2.13}$$

When we substitute $n=0$ in the last equality, we obtain (4.2.6) and consequently it permits to express (4.2.13) as follows:

$$-\big(\mathcal{F}_\varepsilon P_n^{[1]}\big)u_0^{[1]} + 4\Big(x\left(P_n^{[1]}\right)'(u_0^{[1]})\Big)' = A_{n+1}(\cdot;\varepsilon)u_0, \quad n\in\mathbb{N}, \tag{4.2.14}$$

where $A_{n+1}(\cdot;\varepsilon)$ corresponds to the polynomial given by (4.2.11). The particular choice of $n=1$ in (4.2.14) yields

$$-(2+\varepsilon)u_0^{[1]} + 4\Big(xu_0^{[1]}\Big)' = A_2(x)$$

providing (4.2.7), which, in turn, allows to transform (4.2.14) into

$$2\,x\big(P_n^{[1]}(x)\big)''u_0^{[1]} = \Big(A_{n+1}(x) - A_2(x)\big(P_n^{[1]}(x)\big)'\Big)u_0\,, \quad n\in\mathbb{N}. \tag{4.2.15}$$

The particular choice of $n=2$ corresponds to (4.2.8) and enables to write (4.2.15) like

$$\frac{1}{2}\,h(x)\,\big(P_n^{[1]}(x)\big)''u_0 = \Big(A_{n+1}(x) - A_2(x)\big(P_n^{[1]}(x)\big)'\Big)u_0\,, \quad n\in\mathbb{N}$$

which, because of the regularity of $u_0$, provides

$$\frac{1}{2}\,h(x)\,\big(P_n^{[1]}(x)\big)'' - \Big(A_{n+1}(x) - A_2(x)\big(P_n^{[1]}(x)\big)'\Big) = 0\,, \quad n\in\mathbb{N}\,. \tag{4.2.16}$$

By equating the coefficients of the highest degree in this last equality, we figure out

$$n\,(n-1)\,\frac{1}{12}\,\frac{d^3}{dx^3}h(0) - \left(\lambda_n - \lambda_0 - \frac{n}{2}\,\frac{d^2}{dx^2}f(0)\right) = 0,\ n\in\mathbb{N}.$$

Since $\lambda_n \neq 0$ for all $n\in\mathbb{N}$, from the previous we conclude (4.2.12). □

As a consequence of this last result, we present

**Corollary 4.2.3.** *Let $\{P_n\}_{n\in\mathbb{N}}$ be a MOPS with respect to $u_0$. When $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, defined by (4.2.1), is also a MOPS (with respect to $u_0^{[1]}$), then it holds:*

$$(2+\varepsilon)u_0^{[1]} = \big(h(x)u_0\big)' - f(x)u_0 \tag{4.2.17}$$

*where $f(\cdot)$ and $h(\cdot)$ correspond to the two polynomials given by* (4.2.9) *and* (4.2.10), *respectively.*

*Proof.* Upon the assumptions, we have seen in lemma 4.2.2 that the conditions (4.2.7) and (4.2.7)-(4.2.8) hold. The condition (4.2.17) comes as the difference between (4.2.8) after a single differentiation and (4.2.7). □

These last two results are at the basis of the characterisation of the $\mathcal{F}_\varepsilon$-classical sequences.

**Theorem 4.2.4.** *Let $\{P_n\}_{n\in\mathbb{N}}$ be a MOPS with respect to $u_0$. The following statements are equivalent:*

*(a) The MOPS $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{F}_\varepsilon$-classical.*

*(b) The elements of $\{P_n\}_{n\in\mathbb{N}}$ are eigenfunctions of the differential equation*

$$\Big(\mathcal{F}_\varepsilon^*\ \mathcal{F}_\varepsilon P_{n+1}\Big)(x) = 2\lambda_n\rho_{n+1}P_{n+1}(x)\ , \quad n\in\mathbb{N}, \tag{4.2.18}$$

*where*

$$\mathcal{F}_\varepsilon^* = h(x)D^2 + 2\,f(x)\,D + 2\lambda_0 P_1(x)\mathbb{I}\ , \tag{4.2.19}$$

*$\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ represents a sequence of the nonzero numbers given by (4.2.5) and $h, f$ are two polynomials such that $\deg h \leqslant 3$ and $\deg f \leqslant 2$.*

*(c) There exist two polynomials $f$ and $h$, with $\deg f \leqslant 2$ and $\deg h \leqslant 3$, and a nonzero constant $\lambda_0$ satisfying the condition* (4.2.12) *such that the regular form fulfils the two following equations:*

$$\big(h(x)u_0\big)'' - 2\big(f(x)u_0\big)' + 2\lambda_0P_1(x)u_0 = 0 \tag{4.2.20}$$

$$\mathcal{F}_{-\varepsilon}\Big(\big(h(x)u_0\big)' - f(x)u_0\Big) = (2+\varepsilon)\lambda_0P_1(x)u_0 \tag{4.2.21}$$

*(d) There exist two polynomials $f$ and $h$, with $\deg f \leqslant 2$ and $\deg h \leqslant 3$, and a nonzero constant $\lambda_0$ satisfying the condition* (4.2.12) *such that the regular form fulfils simultaneously the equations:*

$$\left\{\left(xf(x) + \frac{2-\varepsilon}{4}h(x)\right)u_0\right\}' - \Big\{2f(x) + 2\,x\,\lambda_0P_1\Big\}u_0 = 0 \tag{4.2.22}$$

$$\mathcal{F}_{-\varepsilon}\Big(h(x)u_0\Big) - 4\Big\{f(x) + \lambda_0\,x\,P_1\Big\}u_0 = 0 \tag{4.2.23}$$

(e) *There exist two polynomials $f$ and $h$, with $\deg f \leqslant 2$ and $\deg h \leqslant 3$, and a nonzero constant $\lambda_0$ satisfying the condition* (4.2.12) *such that the regular form fulfils simultaneously the equations* (4.2.22) *and*

$$\Big\{ xh(x)u_0 \Big\}' - \Big\{ xf(x) + \frac{6+\varepsilon}{4} h(x) \Big\} u_0 = 0 \qquad (4.2.24)$$

*Proof.* The proof will be performed as follows: (a)⇒(b)⇒(c)⇒(a) and afterwards we will show that (c)⇒(d)⇒(e)⇒(c).

The assumption over the $\mathcal{F}_\varepsilon$-classical character of the MOPS $\{P_n\}_{n\in\mathbb{N}}$ corresponds to the assumption of the orthogonality of the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$. Its corresponding regular form will be coherently denoted as $u_0^{[1]}$. Within this context, at the end of the proof of lemma 4.2.2, we have seen that (4.2.16) (under the consideration (4.2.11)) holds. Considering the definition of the polynomial $f$ in (4.2.9), the relation (4.2.16) may be written like

$$\frac{1}{2}\, h(x) \left( P_n^{[1]}(x) \right)'' - \Big\{ \lambda_n P_{n+1}(x) - \lambda_0 P_1(x) P_n^{[1]}(x) - f(x) \, \left( P_n^{[1]}(x) \right)' \Big\} = 0, \quad n \in \mathbb{N},$$

which, by taking $\mathcal{F}_\varepsilon^*$ as in (4.2.19), we get

$$\mathcal{F}_\varepsilon^* P_n^{[1]}(x) = 2\lambda_n \, P_{n+1} \; , \quad n \in \mathbb{N}. \qquad (4.2.25)$$

From the definition of the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ we have $P_n^{[1]}(x) = \frac{1}{\rho_{n+1}} \mathcal{F}_\varepsilon P_{n+1}(x)$, and for this reason the relation (4.2.25) may be transformed into (4.2.18), whence (a) implies (b).

Let us now show that (b) implies (c). Firstly, remark that, under the definition of the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, the relation (4.2.25) comes as a consequence of (4.2.18). For this reason, considering the action of $u_0$ over both sides of (4.2.18) corresponds to perform it over (4.2.25), and we have

$$\left\langle u_0, \left( \mathcal{F}_\varepsilon^* \; P_n^{[1]} \right)(x) \right\rangle = \langle u_0, 2\lambda_n P_{n+1}(x) \rangle \; , \quad n \in \mathbb{N}$$

By duality and because $\langle u_0, P_{n+1}(x) \rangle = 0$, $n \in \mathbb{N}$, the previous equality may be transformed into

$$\left\langle {}^t\mathcal{F}_\varepsilon^* u_0, P_n^{[1]}(x) \right\rangle = 0 \; , \quad n \in \mathbb{N} \; ,$$

Since $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is a MPS (ergo, it spans $\mathcal{P}$), the last identity compels $u_0$ to be such that

$${}^t\mathcal{F}_\varepsilon^* \, u_0 = 0 \; ,$$

which corresponds to (4.2.20).

On the other hand, the action of $\mathcal{F}_\varepsilon$ over both sides of (4.2.25) enables, on account of the definition of $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, the following fourth order linear differential equation fulfilled by the elements of the MPS $\{P_n^{[1]}\}_{n\in\mathbb{N}}$:

$$\left(\mathcal{F}_\varepsilon\;\mathcal{F}_\varepsilon^* P_n^{[1]}\right)(x) = 2\lambda_n\rho_{n+1}P_n^{[1]}(x)\;,\quad n\in\mathbb{N}\;. \tag{4.2.26}$$

The action of the form $u_0^{[1]}$ over both sides of this last equation provides

$$\left\langle u_0^{[1]}, \left(\mathcal{F}_\varepsilon\;\mathcal{F}_\varepsilon^* P_n^{[1]}\right)(x)\right\rangle = 2\lambda_0\rho_1\delta_{n,0}\;,\quad n\in\mathbb{N},$$

which may be transformed into

$$\left\langle {}^t\mathcal{F}_\varepsilon^*\;\mathcal{F}_{-\varepsilon}\left(u_0^{[1]}\right), P_n^{[1]}(x)\right\rangle = 2\lambda_0\rho_1\delta_{n,0}\;,\quad n\in\mathbb{N},$$

and, based on lemma 1.3.1, we deduce

$${}^t\mathcal{F}_\varepsilon^*\;\mathcal{F}_{-\varepsilon}\left(u_0^{[1]}\right) = 2\lambda_0\rho_1\;u_0^{[1]}\;. \tag{4.2.27}$$

As aforementioned, the dual sequences $\{u_n^{[1]}\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are related according to (3.2.3), and in particular we have $\mathcal{F}_{-\varepsilon}\left(u_0^{[1]}\right) = \rho_1 u_1$. The orthogonality of the sequence $\{P_n\}_{n\in\mathbb{N}}$, assures that

$$\mathcal{F}_{-\varepsilon}\left(u_0^{[1]}\right) = \rho_1(\gamma_1)^{-1}P_1(x)u_0\;.$$

Consequently, the relation (4.2.27) becomes

$${}^t\mathcal{F}_\varepsilon^*\left(\rho_1(\gamma_1)^{-1}P_1(x)u_0\right) = 2\lambda_0\rho_1\;u_0^{[1]}\;.$$

The action of $\mathcal{F}_{-\varepsilon}$ on both sides of this last equality leads to

$$\mathcal{F}_{-\varepsilon}\;{}^t\mathcal{F}_\varepsilon^*\left(\rho_1(\gamma_1)^{-1}P_1(x)u_0\right) = 2\lambda_0\rho_1\;\rho_1(\gamma_1)^{-1}P_1(x)u_0\;,$$

*i.e.* ,

$$\mathcal{F}_{-\varepsilon}\;{}^t\mathcal{F}_\varepsilon^*\left(P_1(x)u_0\right) = 2\lambda_0\rho_1P_1(x)u_0\;, \tag{4.2.28}$$

As a matter of fact, we have

$${}^t\mathcal{F}_\varepsilon^*\left(P_1u_0\right) = P_1(x)\Big\{(h(x)u_0)'' - 2(f(x)u_0)' + 2\lambda_0\,P_1(x)u_0\Big\} + 2(h(x)u_0)' - 2f(x)u_0$$

which, after (4.2.20), may be transformed into

$${}^t\mathcal{F}_\varepsilon^*\left(P_1u_0\right) = 2(h(x)u_0)' - 2f(x)u_0\;,$$

enabling (4.2.28) to be transformed into (4.2.21). As a result (b) implies (c).

We show now that (c) implies (a). Supose that $u_0$ fulfils the conditions (4.2.20)-(4.2.21). Let $v$ be a form such that

$$(2+\varepsilon)\,v = \big(h(x)u_0\,\big)' - f(x)u_0 \tag{4.2.29}$$

Thus, the relation (4.2.21) may be read as

$$\mathcal{F}_{-\varepsilon}(v) = \lambda_0 P_1(x)u_0 \tag{4.2.30}$$

According to proposition 1.4.2, the orthogonality of the MPS $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ may be achieved if the conditions $\langle v, x^m P_n^{[1]}\rangle = 0$, for any integer $m$ such that $0 \leqslant m \leqslant n-1$, and $\langle v, x^n P_n^{[1]}\rangle \neq 0$ for any $n\in\mathbb{N}$ hold true.

Following the definition of $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, by transposition of the operator $\mathcal{F}_\varepsilon$, we have

$$\langle v, P_n^{[1]}\rangle = \frac{1}{\rho_{n+1}}\langle \mathcal{F}_{-\varepsilon}(v), P_{n+1}\rangle = \frac{1}{\rho_{n+1}}\,\lambda_0\langle u_0, P_1\,P_{n+1}\rangle\ , \quad n\in\mathbb{N},$$

where the last identity is due to (4.2.30). Consequently, the orthogonality of $\{P_n\}_{n\in\mathbb{N}}$ implies

$$\langle v, P_n^{[1]}\rangle = \frac{1}{\rho_1}\,\lambda_0\delta_{n,0}\ , \quad n\in\mathbb{N}, \tag{4.2.31}$$

Consider $m\in\mathbb{N}^*$ and $n\in\mathbb{N}$, with $1\leqslant m\leqslant n$, recalling the definition of the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, we obtain

$$\langle v, x^m\,P_n^{[1]}\rangle = \frac{1}{\rho_{n+1}}\langle v, x^m\,\mathcal{F}_\varepsilon(P_{n+1})\rangle$$

and following (3.2.1), we are able to write

$$\begin{aligned}
\mathcal{F}_\varepsilon(x^m\,P_{n+1}) &= x^m\mathcal{F}_\varepsilon(P_{n+1}) + \rho_m x^{m-1}\,P_{n+1} + 4\,m\,x^m\,P'_{n+1}\\
&= x^m\mathcal{F}_\varepsilon(P_{n+1}) + 4m\big(x^m\,P_{n+1}\big)' + (\rho_m - 4\,m^2)x^{m-1}P_{n+1}\\
&= x^m\mathcal{F}_\varepsilon(P_{n+1}) + 4m\big(x^m\,P_{n+1}\big)' + (-2\,m+\varepsilon)m\,x^{m-1}P_{n+1},
\end{aligned}$$

allowing us to write

$$\begin{aligned}
\langle v, x^m P_n^{[1]}\rangle = &\ \frac{1}{\rho_{n+1}}\Big\{\ \Big\langle v, \mathcal{F}_\varepsilon(x^m\,P_{n+1}) - 4m\big(x^m\,P_{n+1}\big)'\Big\rangle\\
&- \big\langle v, (-2\,m+\varepsilon)m\,x^{m-1}P_{n+1}\,\big\rangle\Big\}\\
= &\ \frac{1}{\rho_{n+1}}\Big\{\ \langle \mathcal{F}_{-\varepsilon}(v), x^m\,P_{n+1}\rangle + 4\,m\,\big\langle x\,v', x^{m-1}\,P_{n+1}\big\rangle\\
&+ (2\,m-\varepsilon)m\,\big\langle v, x^{m-1}\,P_{n+1}\big\rangle\Big\}\ .\\
= &\ \frac{1}{\rho_{n+1}}\Big\{\ \langle \mathcal{F}_{-\varepsilon}(v), x^m\,P_{n+1}\rangle + m\,\big\langle 4\,x\,v' + (2-\varepsilon)v, x^{m-1}\,P_{n+1}\big\rangle\\
&+ 2(m-1)\,\big\langle v, x^{m-1}\,P_{n+1}\big\rangle\Big\}\ .
\end{aligned} \tag{4.2.32}$$

According to (4.2.20), we have

$$(2+\varepsilon)v' = \big(f(x)u_0\big)' - 2\lambda_0 P_1(x)u_0 \tag{4.2.33}$$

Based on this last equality, (4.2.30) becomes

$$2\mathcal{F}_{-\varepsilon}(v) = \big(f(x)u_0\big)' - (2+\varepsilon)v' \;,$$

that is,

$$2\big(2x\,v' - \varepsilon\,v\big)' = \big(f(x)u_0 - (2+\varepsilon)v\big)' \;,$$

and due to the injectivity of the differential operator on $\mathcal{P}'$, we deduce

$$4x\,v' + (2-\varepsilon)v = f(x)u_0 \;. \tag{4.2.34}$$

Recalling the definition of $v$ in (4.2.29), we replace in this last identity the term in $u_0$ and we obtain

$$4x\,v' + 4v = \big(h(x)u_0\big)'$$

yielding

$$4\,x\,v = h(x)u_0 \;. \tag{4.2.35}$$

By virtue of (4.2.30), (4.2.34) and (4.2.35), for $m,n \in \mathbb{N}^*$ with $1 \leqslant m \leqslant n$, the first and last members of (4.2.32) equate to

$$\begin{aligned}\langle v\,,\,x^m P_n^{[1]}\rangle \;=\;& \frac{1}{\rho_{n+1}}\Big\{ \big\langle \lambda_0 P_1(x)\,u_0 + m\,f(x)\,u_0\,,\,x^{m-1}\,P_{n+1}\big\rangle \\ &+\frac{m(m-1)}{2}\big\langle h(x)\,u_0\,,\,x^{m-2}\,P_{n+1}\big\rangle\Big\}\end{aligned}$$

This latter and (4.2.31) may be resumed as

$$\begin{aligned}\langle v\,,\,x^m P_n^{[1]}\rangle \;=\;& \frac{1}{\rho_{n+1}}\Big\{ \langle u_0\,,\,\lambda_0 P_1(x)x^m P_{n+1}\rangle + m\,\langle u_0\,,\,f(x)x^m\,P_{n+1}\rangle \\ &+\frac{m(m-1)}{2}\big\langle\,u_0\,,\,x^{m-2}h(x)\,P_{n+1}\big\rangle\Big\}\,, \quad 0 \leqslant m \leqslant n.\end{aligned}$$

Since $\deg f \leqslant 2$, $\deg h \leqslant 3$, then according to proposition 1.4.2, the orthogonality of $\{P_n\}_{n\in\mathbb{N}}$ guarantees

$$\langle v, x^m P_n^{[1]}\rangle = 0\;, \quad 0 \leqslant m \leqslant n-1,\; n \geqslant 1,$$

and when $m = n$ with $n \in \mathbb{N}$, we get

$$\langle v, x^n P_n^{[1]}\rangle = \frac{1}{\rho_{n+1}}\left\{\lambda_0 + \frac{n}{2}\,f''(0) + n(n-1)\frac{1}{12}\frac{d^3}{dx^3}h(0)\right\}\,\big\langle u_0, x^{n+1}\,P_{n+1}\big\rangle \neq 0$$

because of (4.2.12). As a result, again from proposition 1.4.2, we conclude that $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is necessarily orthogonal with respect to $v$, whence $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{F}_\varepsilon$-classical.

Let us show that (c) implies (d). Between (4.2.20)-(4.2.21) it is possible to eliminate the term in the right hand side of (4.2.21) and this leads to

$$2\,\mathcal{F}_{-\varepsilon}\Big((\,h(x)u_0\,)' - f(x)u_0\Big) = (2+\varepsilon)\Big\{-(\,h(x)u_0\,)' + 2f(x)u_0\Big\}'$$

*i.e.* ,

$$\begin{aligned}&\bigg(2\Big\{2x(\,h(x)u_0\,)'' - 2x(\,f(x)u_0\,)' - \varepsilon(\,h(x)u_0\,)' + \varepsilon\, f(x)u_0\Big\}\bigg)'\\&= (2+\varepsilon)\bigg(-(\,h(x)u_0\,)' + 2\,f(x)u_0\bigg)' .\end{aligned}$$

The injectivity of the derivative operator in $\mathcal{P}'$ permits to obtain from the previous identity

$$\begin{aligned}&2\Big\{2x(\,h(x)u_0\,)'' - 2x(\,f(x)u_0\,)' - \varepsilon(\,h(x)u_0\,)' + \varepsilon\, f(x)u_0\Big\}\\&= (2+\varepsilon)\bigg(-(\,h(x)u_0\,)' + 2\,f(x)u_0\bigg) ,\end{aligned}$$

*i.e.* ,

$$4x\Big\{(\,h(x)u_0\,)'' - (\,f(x)u_0\,)'\Big\} + (2-\varepsilon)\,(\,h(x)u_0\,)' - 4\,f(x)u_0 = 0 .$$

Now, on account of (4.2.20), the previous may be transformed into the following equality

$$4x\Big\{(\,f(x)u_0\,)' - 2\lambda_0 P_1 u_0\Big\} + (2-\varepsilon)(\,h(x)u_0\,)' - 4f(x)u_0 = 0$$

yielding (4.2.22). Besides, the multiplication of the equation (4.2.20) by $(2x)$ furnishes

$$2x\,\Big(h(x)u_0\Big)'' - 4x\,\Big(f(x)u_0\Big)' + 4\lambda_0\,x\,P_1(x)\,u_0 = 0$$

which, due to the definition of $\mathcal{F}_{-\varepsilon}$, may be rewritten like

$$\mathcal{F}_{-\varepsilon}\Big(h(x)u_0\Big) - (2-\varepsilon)\Big(h(x)u_0\Big)' - 4x\,\Big(f(x)u_0\Big)' + 4\lambda_0\,xP_1(x)u_0 = 0$$

*i.e.* ,

$$\mathcal{F}_{-\varepsilon}\Big(h(x)u_0\Big) - \Big(\{4xf(x) + (2-\varepsilon)h(x)\}u_0\Big)' + 4\Big\{f(x) + \lambda_0\,xP_1(x)\Big\}u_0 = 0 .$$

By virtue of (4.2.22), this last enables (4.2.23). Whence (c)⇒(d).

Let us prove that (d) implies (e). On account of (4.2.22), from (4.2.23) we deduce

$$\mathcal{F}_{-\varepsilon}\Big(h(x)u_0\Big) - 2\Big(\{xf(x) + \frac{2-\varepsilon}{4}h(x)\}u_0\Big)' = 0$$

which corresponds to

$$\left(2x\Big(h(x)u_0\Big)' - \left\{\varepsilon\, h(x) + 2xf(x) + \frac{2-\varepsilon}{2}h(x)\right\} u_0\right)' = 0\ .$$

The injectivity of the derivative in $\mathcal{P}'$ allows to deduce

$$2x\Big(h(x)u_0\Big)' + \left\{2xf(x) + \frac{2+\varepsilon}{2}h(x)\right\} u_0 = 0\,,$$

which provides (4.2.24), once we have $2x\big(h(x)u_0\big)' = \big(2x\,h(x)u_0\big)' - 2h(x)u_0$. This ends the proof that (d) implies (e).

Finally, we show that (e) implies (c).

After a single differentiation, the equation (4.2.24) becomes

$$\Big(x\ \big(h(x)u_0\big)'\Big)' - \Big(\big(xf(x) + \frac{2+\varepsilon}{4}h(x)\big)u_0\Big)' = 0$$

which, on account of the definition of the operator $\mathcal{F}_\varepsilon$, may be transformed into

$$\frac{1}{2}\,\mathcal{F}_{-\varepsilon}\big(h(x)u_0\big) - \left\{\left(xf(x) + \frac{2-\varepsilon}{4}h(x)\right) u_0\right\}' = 0\ .$$

Due to (4.2.22), we are able to rewrite this last equality as follows

$$\frac{1}{2}\,\mathcal{F}_{-\varepsilon}\big(h(x)u_0\big) - 2\left\{\left(xf(x) + \frac{2-\varepsilon}{4}h(x)\right) u_0\right\}' + 2\Big\{f(x) + \lambda_0\, x\, P_1(x)\Big\}u_0 = 0$$

which, by recalling the definition of the operator $\mathcal{F}_{-\varepsilon}$, may be expressed like

$$x\,\Big\{\big(h(x)u_0\big)'' - 2\big(f(x)u_0\big)' + \lambda_0\, P_1(x)\Big\} = 0\,.$$

Dividing the last equation by $x$, in accordance with (1.2.6), we obtain

$$\big(h(x)u_0\big)'' - 2\big(f(x)u_0\big)' + \lambda_0\, P_1(x) + \big\langle\big(h(x)u_0\big)'' - 2\big(f(x)u_0\big)' + \lambda_0\, P_1(x), 1\big\rangle\delta = 0\,.$$

which brings (4.2.20) because of the orthogonality of $\{P_n\}_{n\in\mathbb{N}}$. On the other hand, we have

$$\begin{aligned}&\mathcal{F}_{-\varepsilon}\Big(\big(h(x)u_0\big)' - f(x)u_0\Big)\\ &= 2\Big(x\big(h(x)u_0\big)'' - x\big(f(x)u_0\big)'\Big)' - \varepsilon\Big(\big(h(x)u_0\big)'' - \big(f(x)u_0\big)'\Big)\end{aligned} \tag{4.2.36}$$

Insofar as the equality (4.2.20) is assured, we may write

$$\big(h(x)u_0\big)'' - \big(f(x)u_0\big)' = \big(f(x)u_0\big)' - 2\lambda_0 P_1(x)u_0$$

while, by taken into account (4.2.22), we have

$$\begin{aligned} x\big(h(x)u_0\big)'' - x\big(f(x)u_0\big)' &= x\big(f(x)u_0\big)' - 2x\lambda_0 P_1(x)u_0 \\ &= -\frac{2-\varepsilon}{4}\big(h(x)u_0\big)' + f(x)u_0 \end{aligned}$$

As a result, (4.2.36) becomes

$$\begin{aligned} &\mathcal{F}_{-\varepsilon}\Big(\big(h(x)u_0\big)' - f(x)u_0\Big) \\ &= \Big(-\frac{2-\varepsilon}{2}\big(h(x)u_0\big)' + 2f(x)u_0\Big)' - \varepsilon\Big\{\big(f(x)u_0\big)' - 2\lambda_0 P_1(x)u_0\Big\} \end{aligned}$$

Once more, because of (4.2.20) this latter yields

$$\begin{aligned} &\mathcal{F}_{-\varepsilon}\Big(\big(h(x)u_0\big)' - f(x)u_0\Big) \\ &= \Big(-\frac{2-\varepsilon}{2}\Big[2\big(f(x)u_0\big)' - 2\lambda_0 P_1 u_0\Big] + 2\big(f(x)u_0\big)'\Big) - \varepsilon\Big\{\big(f(x)u_0\big)' - 2\lambda_0 P_1(x)u_0\Big\} \end{aligned}$$

which corresponds to (4.2.21) and, consequently, we conclude the proof. □

The previous result provides a characterisation of the $\mathcal{F}_\varepsilon$-classical forms (and therefore the $\mathcal{F}_\varepsilon$-classical sequences). Particularly, from the equation (4.2.22) of statement (d) we may read a $\mathcal{F}_\varepsilon$-classical form as a semiclassical form of class $s \leqslant 1$. In order to infer about the class of the semiclassical form $u_0$, we could now opt for using lemma 2.3.2 (p. 71), since statement (e) of theorem 4.2.4 assures that $u_0$ fulfils both (4.2.22) and (4.2.24). However, the complexity on the expressions for $h(\cdot)$ or $f(\cdot)$ turns this problem quite tricky to solve. Therefore other process will be behind the resolution of this problem, as it will be expounded in section 4.2.3.

Apart from the existent condition relating the two sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ furnished by the definition of the last one, we have obtained (4.2.25) which provided two fourth order differential equations: one fulfilled by the elements of $\{P_n\}_{n\in\mathbb{N}}$ and other by those of $\{P_n^{[1]}\}_{n\in\mathbb{N}}$. By expanding the operators $\mathcal{F}_\varepsilon\mathcal{F}_\varepsilon^*$ and $\mathcal{F}_\varepsilon^*\mathcal{F}_\varepsilon$ in terms of the derivative operator, we obtain

**Corollary 4.2.5.** *If the MOPS $\{P_n\}_{n\in\mathbb{N}}$ is such that $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ given by (4.2.1) is a MOPS, then each element of $\{P_n\}_{n\in\mathbb{N}}$ fulfils the forth order differential equation*

$$\begin{aligned} &xh(x)P_{n+1}^{(4)}(x) + \Big\{\frac{1}{2}(6+\varepsilon)\,h(x) + 2x\,f(x)\Big\}P_{n+1}^{(3)}(x) \\ &+\Big\{(4+\varepsilon)\,f(x) + 2\lambda_0 x P_1(x)\Big\}P_{n+1}''(x) + (2+\varepsilon)\lambda_0 P_1(x)\,P_{n+1}'(x) \\ &= (n+1)\big(2(n+1)+\varepsilon\big)\,\lambda_n(\varepsilon)\,P_{n+1}(x)\,,\, n\in\mathbb{N}\,. \end{aligned} \tag{4.2.37}$$

*while the elements of* $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ *fulfil*

$$\begin{aligned}
&x\,h(x)\,\Big(P_n^{[1]}\Big)^{(4)}(x) + \Big(\frac{2+\varepsilon}{2}h(x) + 2x\,(f(x)+h'(x))\Big)\,\Big(P_n^{[1]}\Big)^{(3)}(x)\\
&+\Big(4xf'(x) + \frac{2+\varepsilon}{2}\,(2f(x)+h'(x)) + xh''(x)\Big)\,\Big(P_n^{[1]}\Big)''(x)\\
&+\Big((2+\varepsilon)f'(x) + 2xf''(x)\Big)\,\Big(P_n^{[1]}\Big)'(x)\\
&= \Big(\rho_{n+1}\,\lambda_n - (2+\varepsilon)\lambda_0\Big)P_n^{[1]}(x)\,,\quad n\in\mathbb{N}.
\end{aligned}$$

*Proof.* It is a mere consequence of (4.2.18) and (4.2.26). □

The differential equations just found are of even order with polynomial coefficients not depending on $n$. We wished to say that such condition would be sufficient to conclude that $\{P_n\}_{n\in\mathbb{N}}$ is $D$-classical, leaving aside the possibility of $\{P_n\}_{n\in\mathbb{N}}$ being a semiclassical sequence of class 1. Unfortunately, this is not possible. Even theorem 2.4.1 is useless here.

On the other hand, in 1940 Krall [63] sought to determine all the orthogonal polynomial solutions, with respect to some (possibly signed) Borel measure on the Borel subsets of the real line, of a fourth order differential equation of the type (2.0.7) with $N=4$. Krall determined the contents of the polynomial solutions of such problem, up to a linear change of variable, by an exhaustive method involving over forty cases. However, he was looking for PSs orthogonal with respect to certain measures and disregard any polynomial sequences that did not possess this type of orthogonality. His work was followed by other authors, who attempted to discover orthogonal polynomial solutions of such differential equation. Among them we quote the works of Everitt and Littlejohn [40], Everitt et al. [41, 42], Kwon and Yoon [68]. Hitherto, no complete classification of the polynomial solutions of this problem is available.

We will keep searching for the $\mathcal{F}_\varepsilon$-classical polynomials, based on the theory of moment forms developed by Maroni, instead of fishing for already analysed cases in the quoted works.

### 4.2.2 About the invariance of the $\mathcal{F}_\varepsilon$-classical character by a linear transformation

As previously pointed out on the example of page 30, the MPS $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$ defined by

$$\widetilde{P}_n(x) = a^{-n}P_n(ax+b),\ \text{with}\ a\in\mathbb{C}^*, b\in\mathbb{C},\ n\in\mathbb{N}.$$

is orthogonal with respect to the form $\widetilde{u}_0 = \big(h_{a^{-1}}\circ\tau_{-b}\big)u_0$ as long as $\{P_n\}_{n\in\mathbb{N}}$ is a MOPS with respect to $u_0$.

Next, we will be working out to show that the fact that the MOPS $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{F}_\varepsilon$-classical provides the $\mathcal{F}_\varepsilon$-classical character of the MOPS $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$.

Following the considerations made on page 70 concerned with the invariance of the semi-classical character, if the regular form $u_0$ fulfils both (4.2.22) and (4.2.24), then the regular form $\widetilde{u}_0$ satisfies

$$\left(\widetilde{\Phi}_1(x)\,\widetilde{u}_0\right)' + \widetilde{\Psi}_1(x)\widetilde{u}_0 = 0$$
$$\left(\widetilde{\Phi}_2(x)\,\widetilde{u}_0\right)' + \widetilde{\Psi}_2(x)\widetilde{u}_0 = 0$$

with

$$\widetilde{\Phi}_i(x) = a^{-\deg\Phi_i}\,\Phi_i(ax+b) \quad ; \quad \widetilde{\Psi}_i(x) = a^{1-\deg\Phi_i}\,\Psi_i(ax+b)\ ,\ i=1,2,$$

where

$$\Phi_1(x) = xf(x) + \tfrac{2-\varepsilon}{4}h(x) \quad , \quad \Psi_1(x) = -2f(x) - 2\lambda_0 xP_1(x)$$
$$\Phi_1(x) = xh(x) \quad , \quad \Psi_1(x) = -xf(x) - \tfrac{6+\varepsilon}{4}h(x)$$

Upon this and considering theorem 4.2.4, comes the conclusion: the MOPS $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$ is $\mathcal{F}_\varepsilon$-classical, as long as the MOPS $\{P_n\}_{n\in\mathbb{N}}$ is.

### 4.2.3 Construction of the $\mathcal{F}_\varepsilon$-classical sequences

The techniques used here are essentially constructive. The sketch of this construction is based either on (4.2.18) (which equates (4.2.37)) or on (4.2.22)-(4.2.24). If, on one hand (4.2.18) provides relations over the recurrence coefficients of the two MOPSs $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, on the other hand, between the functional differential relations (4.2.22)-(4.2.24) fulfilled by any $\mathcal{F}_\varepsilon$-classical form $u_0$ it is possible, as we will see, to figure out a necessary condition to be fulfilled by the polynomials $f(\cdot)$, $h(\cdot)$ and $\lambda_0 P_1(\cdot)$. Henceforth, we will be dealing with the resolution of a nonlinear system of equations that will furnish the expressions for the polynomials $h(\cdot)$ and $f(\cdot)$ and concomitantly, the expressions for the recurrence coefficients associated to $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$.

#### 4.2.3.1 Relations satisfied by the recurrence coefficients

So far, we have given differential relations over the regular form $u_0$ and also over the elements of the two MOPSs $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$. Based on the achieved results, we now aim to obtain more information about the recurrence coefficients $(\beta_n, \gamma_{n+1})_{n\in\mathbb{N}}$ and $(\beta_n^{[1]}, \gamma_{n+1}^{[1]})_{n\in\mathbb{N}}$

associated to the second order recurrence relations fulfilled by the two MOPSs $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$:

$$\begin{cases} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0 \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x)\,, \quad n \in \mathbb{N}, \end{cases} \tag{4.2.38}$$

and

$$\begin{cases} P_0^{[1]}(x) = 1 \quad , \quad P_1^{[1]}(x) = x - \beta_0^{[1]} \\ P_{n+2}^{[1]}(x) = (x-\beta_{n+1}^{[1]})P_{n+1}^{[1]}(x) - \gamma_{n+1}^{[1]}P_n^{[1]}(x)\,, \quad n \in \mathbb{N}, \end{cases} \tag{4.2.39}$$

with $\gamma_{n+1}\,,\, \gamma_{n+1}^{[1]} \neq 0,\, n\in\mathbb{N}$.

The determination of such recurrence coefficients lies, at least in part, in the differential equation (4.2.37) or, equivalently, in the two known relations between the elements of both sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ here numbered as (4.2.1) and (4.2.25). However this is not easy to accomplish if we use the recurrence relations directly, as in the procedure taken in the resolution of the analogous problem in the study of the $(D)$-classical sequences (see Maroni [78]). So one may instead expand the sequences $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ in powers of the variable $x$ and then make the comparison in both relations (4.2.1) and (4.2.25) of, at least, the three terms in the highest powers. In other words, we transfer the determination of the recurrence coefficients of the MOPS $\{P_n\}_{n\in\mathbb{N}}$ to the determination of its three highest powers coefficients of its variable. In order to set this in concrete, let us represent the elements of $\{P_n\}_{n\in\mathbb{N}}$ and $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ as follows:

$$P_n(x) = x^n + b_n\, x^{n-1} \; + c_{n-1}\, x^{n-2} + \ldots \tag{4.2.40}$$

$$P_n^{[1]}(x) = x^n + b_n^{[1]}\, x^{n-1} + c_{n-1}^{[1]}\, x^{n-2} + \ldots \tag{4.2.41}$$

with $b_0 = c_0 = c_{-1} = b_0^{[1]} = c_0^{[1]} = c_{-1}^{[1]} = 0$.

The use of the previous equalities in the relations (4.2.38) and (4.2.39), respectively, provides $b_1 = -\beta_0$ and

$$\begin{cases} b_{n+1} = b_n - \beta_n \\ c_{n+1} = c_n - \beta_{n+1}b_{n+1} - \gamma_{n+1} \end{cases} \qquad n\in\mathbb{N}, \tag{4.2.42}$$

$$\begin{cases} b_{n+1}^{[1]} = b_n^{[1]} - \beta_n^{[1]} \\ c_{n+1}^{[1]} = c_n^{[1]} - \beta_{n+1}^{[1]}b_{n+1}^{[1]} - \gamma_{n+1}^{[1]} \end{cases} \qquad n\in\mathbb{N}. \tag{4.2.43}$$

or, equivalently,

$$\begin{cases} \beta_n = b_n - b_{n+1} \\ \gamma_{n+1} = c_n - c_{n+1} - (b_{n+1} - b_{n+2})b_{n+1} \end{cases} \qquad n\in\mathbb{N}, \tag{4.2.44}$$

$$\begin{cases} \beta_n^{[1]} = b_n^{[1]} - b_{n+1}^{[1]} \\ \gamma_{n+1}^{[1]} = c_n^{[1]} - c_{n+1}^{[1]} - (b_{n+1}^{[1]} - b_{n+2}^{[1]})b_{n+1}^{[1]} \end{cases} \qquad n\in\mathbb{N}. \tag{4.2.45}$$

On the other hand, replacing in (4.2.1) $P_{n+1}$ and $P_n^{[1]}$ by the corresponding expressions given by (4.2.40) and (4.2.41) and then equating the coefficients in the powers of $x$ leads to

$$\begin{cases} \rho_{n+1} b_n^{[1]} = \rho_n b_{n+1}, & n \in \mathbb{N} \\ \rho_{n+2} c_n^{[1]} = \rho_n c_{n+1}, & n \in \mathbb{N}. \end{cases} \tag{4.2.46}$$

with

$$\rho_{n+1} = (n+1)\big(2(n+1)+\varepsilon\big) \qquad n \in \mathbb{N}.$$

We are now ready to infer the expressions of the first coefficients of $P_{n+1}$ and $P_n^{[1]}$. Considering the expansion of the polynomials $h$ and $f$

$$h(x) = h_3x^3 + h_2x^2 + h_1x + h_0 \qquad ; \qquad f(x) = f_2x^2 + f_1x + f_0, \tag{4.2.47}$$

we replace in (4.2.25) the polynomials $P_n$ and $P_n^{[1]}$ by their corresponding expressions, stated in (4.2.40) and (4.2.41), and this leads to

$$\begin{aligned}
&2\lambda_n(\varepsilon)\big\{x^{n+1} + b_{n+1}x^n + c_nx^{n-1} + \dots\big\} \\
&= \big\{h_3x^3 + h_2x^2 + h_1x + h_0\big\}\Big\{n(n-1)x^{n-2} + (n-1)(n-2)b_n^{[1]}x^{n-3} \\
&+(n-2)(n-3)c_{n-1}^{[1]}x^{n-4} + \dots\Big\} \\
&+2\left\{f_2x^2 + f_1x + f_0\right\}\Big\{nx^{n-1} + (n-1)b_n^{[1]}x^{n-2} + (n-2)c_{n-1}^{[1]}x^{n-3} + \dots\Big\} \\
&+2\lambda_0\big(x+b_1\big)\Big\{x^n + b_n^{[1]}x^{n-1} + c_{n-1}^{[1]}x^{n-2} + \dots\Big\},\ n \in \mathbb{N},
\end{aligned}$$

Equating the coefficients in $x^{n+1}, x^n, x^{n-1}$ in the previous relation, we obtain for any $n \in \mathbb{N}$ the following conditions:

$$\begin{aligned}
2\ \lambda_n &= h_3\, n(n-1) + 2f_2\, n + 2\lambda_0 \\
2\ \lambda_n\ b_{n+1} &= b_n^{[1]}\ \Big\{h_3\ (n-1)(n-2) + 2f_2\ (n-1) + 2\lambda_0\Big\} \\
&\quad + (n-1)n\ h_2 + 2nf_1 + 2\lambda_0 b_1 \\
2\ \lambda_n\ c_n &= c_{n-1}^{[1]}\ \Big\{h_3\ (n-3)(n-2) + 2f_2\ (n-2) + 2\lambda_0\Big\} \\
&\quad + b_n^{[1]}\ \Big\{(n-2)(n-1)\ h_2 + 2(n-1)f_1 + 2\lambda_0 b_1\Big\} \\
&\quad + 2nf_0 + n(n-1)h_1.
\end{aligned}$$

By virtue of (4.2.46) the previous relations may be transformed into the following ones:

$$\begin{aligned}
2\,\lambda_n &= h_3\,n(n-1) + 2f_2\,n + 2\lambda_0 \\
2\,\lambda_n\,b_{n+1} &= \frac{\rho_n}{\rho_{n+1}} b_{n+1}\,\Big\{h_3\,(n-1)(n-2) + 2f_2\,(n-1) + 2\lambda_0\Big\} \\
&\quad + (n-1)n\,h_2 + 2nf_1 + 2\lambda_0 b_1 \\
2\,\lambda_n\,c_n &= \frac{\rho_{n-1}}{\rho_{n+1}} c_n\,\Big\{h_3\,(n-3)(n-2) + 2f_2\,(n-2) + 2\lambda_0\Big\} \\
&\quad + \frac{\rho_n}{\rho_{n+1}} b_{n+1}\,\Big\{(n-2)(n-1)\,h_2 + 2(n-1)f_1 + 2\lambda_0 b_1\Big\} \\
&\quad + 2nf_0 + n(n-1)h_1, \quad n \in \mathbb{N},
\end{aligned}$$

with the convention $\rho_{-1} = \rho_0 = 0$. As a result, we get

$$2\,\lambda_n = h_3\,n(n-1) + 2f_2\,n + 2\lambda_0 \tag{4.2.48}$$

$$2\,\lambda_n\,b_{n+1} = 2\lambda_{n-1}\,\frac{\rho_n}{\rho_{n+1}} b_{n+1}\; + (n-1)n\,h_2 + 2nf_1 + 2\lambda_0 b_1 \tag{4.2.49}$$

$$\begin{aligned}
2\,\lambda_n\,c_n &= 2\lambda_{n-2}\,\frac{\rho_{n-1}}{\rho_{n+1}} c_n + \frac{\rho_n}{\rho_{n+1}} b_{n+1}\,\Big\{(n-2)(n-1)\,h_2 \\
&\quad + 2(n-1)f_1 + 2\lambda_0 b_1\Big\} + 2nf_0 + n(n-1)h_1, \quad n \in \mathbb{N},
\end{aligned} \tag{4.2.50}$$

with the convention $\lambda_{-1} = \lambda_{-2} = 0$. When $n = 0$, the relations (4.2.49) and (4.2.50) are identically satisfied.

To avoid the use of negative or null indexes on any of the relations (4.2.49) and (4.2.50), we shall take $n \to n+1$ in (4.2.49) and $n \to n+2$ in (4.2.50), and we have

$$\lambda_n = \frac{1}{2} h_3\,n(n-1) + f_2\,n + \lambda_0 \tag{4.2.51}$$

$$\Big(\lambda_{n+1}\rho_{n+2} - \lambda_n\rho_{n+1}\Big) b_{n+2} = \rho_{n+2}\Big\{\frac{n\,(n+1)}{2}\,h_2 + (n+1)f_1 + \lambda_0 b_1\Big\} \tag{4.2.52}$$

$$\begin{aligned}
\Big(\lambda_{n+2}\rho_{n+3} - \lambda_n\rho_{n+1}\Big) c_{n+2} &= \rho_{n+2} b_{n+3}\Big\{\frac{(n+1)n}{2}\,h_2 + (n+1)f_1 + \lambda_0 b_1\Big\} \\
&\quad + \rho_{n+3}\Big\{(n+2)f_0 + \frac{1}{2}(n+2)(n+1)\,h_1\Big\},
\end{aligned} \tag{4.2.53}$$

$$c_1 = \frac{\rho_1\,\lambda_0}{\rho_2\,\lambda_1}\,b_2 b_1 + \frac{1}{\lambda_1}\,f_0 \tag{4.2.54}$$

for any $n \in \mathbb{N}$.

Following the expression of $\lambda_n$ in terms of $h_3, f_2$ and $\lambda_0$ provided by (4.2.51), we have[2]

$$\begin{aligned}\lambda_{n+1}\rho_{n+2} - \lambda_n\ \rho_{n+1} = &\ (4n+\varepsilon+6)\lambda_0 + 2(n+1)(3n+\varepsilon+4)f_2 \\ &+\frac{1}{2}n(n+1)(8n+3\varepsilon+10)h_3\end{aligned} \tag{4.2.55}$$

and also

$$\begin{aligned}\lambda_{n+2}\rho_{n+3} - \lambda_n\ \rho_{n+1} = &\ 2(4n+\varepsilon+8)\lambda_0 \\ &+\big[6(\varepsilon+6)+4n(3n+\varepsilon+10)\big]f_2 \\ &+(n+1)\big[3(\varepsilon+6)+n(8n+3\varepsilon+22)\big]h_3\end{aligned} \tag{4.2.56}$$

**Lemma 4.2.6.** *The coefficients* $(b_n, c_{n+1})_{n\in\mathbb{N}}$ *are related by*

$$\lambda_0\Big(-c_1-(b_1-b_2)b_1\Big) = 2+\varepsilon \tag{4.2.57}$$

*and*

$$\begin{aligned}&\Big(\lambda_{n+1}-\lambda_n\frac{\rho_n}{\rho_{n+1}}\Big)\ c_{n+1} - \Big(\lambda_{n+1}-\lambda_n\frac{\rho_{n+2}}{\rho_{n+3}}\Big)\ c_{n+2} \\ &= b_{n+2}\Big\{\Big(\lambda_{n+1}-\lambda_n\frac{\rho_{n+1}}{\rho_{n+2}}\Big)\ b_{n+2} - \Big(\lambda_{n+1}-\lambda_n\frac{\rho_{n+2}}{\rho_{n+3}}\Big)\ b_{n+3}\Big\},\ n\in\mathbb{N}\,.\end{aligned} \tag{4.2.58}$$

*Proof.* We recall that $\gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle}$ and $\gamma_{n+1}^{[1]} = \frac{\langle u_0, (P_{n+1}^{[1]})^2\rangle}{\langle u_0, (P_n^{[1]})^2\rangle}$ for $n\in\mathbb{N}$. From the definition of $\lambda_n$ (4.2.5) we may read

$$\lambda_0\ \gamma_1 = 2+\varepsilon \tag{4.2.59}$$

$$\rho_{n+1}\ \lambda_{n+1}\ \gamma_{n+2} = \rho_{n+2}\ \lambda_n\ \gamma_{n+1}^{[1]}\,, \qquad n\in\mathbb{N}\,. \tag{4.2.60}$$

The relation (4.2.44) with $n=0$ permits to write $\gamma_1 = -c_1-(b_1-b_2)b_1$. Thus, a simple replacement of the precedent expression for $\gamma_1$ in the relation (4.2.59) produces (4.2.57). On the other hand, by replacing in (4.2.60) $\gamma_{n+2}$ and $\gamma_{n+1}^{[1]}$ by their corresponding expressions given in (4.2.44) and (4.2.45), we get

$$\begin{aligned}&\rho_{n+1}\ \lambda_{n+1}\ \Big(c_{n+1}-c_{n+2}-(b_{n+2}-b_{n+3})b_{n+2}\Big) \\ &= \rho_{n+2}\ \lambda_n\ \Big(c_n^{[1]}-c_{n+1}^{[1]}-(b_{n+1}^{[1]}-b_{n+2}^{[1]})b_{n+1}^{[1]}\Big), \quad n\in\mathbb{N}\,,\end{aligned}$$

which, on account of (4.2.46), becomes

$$\begin{aligned}&\rho_{n+1}\ \lambda_{n+1}\ \Big(c_{n+1}-c_{n+2}-(b_{n+2}-b_{n+3})b_{n+2}\Big) \\ &= \rho_{n+2}\ \lambda_n\ \left\{\frac{\rho_n}{\rho_{n+2}}c_{n+1} - \frac{\rho_{n+1}}{\rho_{n+3}}c_{n+2} - \Big(\frac{\rho_{n+1}}{\rho_{n+2}}b_{n+2} - \frac{\rho_{n+2}}{\rho_{n+3}}b_{n+3}\Big)\frac{\rho_{n+1}}{\rho_{n+2}}b_{n+2}\right\},\ n\in\mathbb{N}\,,\end{aligned}$$

[2]The relations (4.2.55) and (4.2.56) were verified in *Mathematica 6.0*©

that is

$$\begin{aligned}&\lambda_{n+1}\Big(c_{n+1}-c_{n+2}-(b_{n+2}-b_{n+3})b_{n+2}\Big)\\ &=\ \lambda_n\ \left\{\frac{\rho_n}{\rho_{n+1}}c_{n+1}-\frac{\rho_{n+2}}{\rho_{n+3}}c_{n+2}-\left(\frac{\rho_{n+1}}{\rho_{n+2}}b_{n+2}-\frac{\rho_{n+2}}{\rho_{n+3}}b_{n+3}\right)b_{n+2}\right\},\quad n\in\mathbb{N}\,,\end{aligned}$$

which corresponds to (4.2.58) after rearranging the order of the terms. □

So far, we have expressed the recurrence coefficients $(\beta_n,\gamma_{n+1})_{n\in\mathbb{N}}$ in terms of the coefficients $(b_n,c_n)_{n\in\mathbb{N}}$ which, in turn, depend on the coefficients of the polynomials $f(\cdot)$ and $h(\cdot)$. Therefore, the conditions (4.2.58) intrinsically gather information about the conditions that $f_i$ and $h_{i+1}$ (with $i=0,1,2$) must satisfy. The determination of these coefficients is of major importance for obvious reasons. However, from (4.2.58) it is hard to decipher expressions for $f_i$ and $h_{i+1}$ (with $i=0,1,2$), because of the inherent large relations involved. In order to overcome this problem we may take into account the statement (b) of theorem 4.2.4, enabling to obtain the following result.

**Lemma 4.2.7.** *Let $u_0$ be a $\mathcal{F}_\varepsilon$-classical form. The polynomials $h(\cdot)$and $f(\cdot)$ fulfilling* (4.2.22)-(4.2.24) *obey to the following condition:*

$$\begin{aligned}&x^2\,f(x)\Big\{h'(x)-f(x)\Big\}\\ &=h(x)\Big\{x^2f'(x)+\frac{(2-\varepsilon)(2+\varepsilon)}{16}\,h(x)-2\lambda_0x^2P_1\Big\}\end{aligned}\tag{4.2.61}$$

*with* $\deg f\leqslant 2$ *and* $\deg h\leqslant 3$.

*Proof.* Suppose $\{P_n\}$ From (4.2.22) and (4.2.24) we respectively get

$$\left(xf(x)+\frac{2-\varepsilon}{4}h(x)\right)u_0'+\left\{xf'(x)+\frac{2-\varepsilon}{4}h'(x)-f(x)-2\,x\,\lambda_0(\varepsilon)P_1\right\}u_0=0$$

$$xh(x)\,u_0'+\left\{xh'(x)-xf(x)-\frac{2+\varepsilon}{4}h(x)\right\}u_0=0\,,$$

Between the two previous relations, we proceed to the elimination of $u_0'$ and this provides the single condition

$$\begin{aligned}&\left(xf(x)+\frac{2-\varepsilon}{4}h(x)\right)\left\{xh'(x)-xf(x)-\frac{2+\varepsilon}{4}h(x)\right\}u_0\\ &=xh(x)\Big\{xf'(x)+\frac{2-\varepsilon}{4}h'(x)-f(x)-2\,x\,\lambda_0(\varepsilon)P_1\Big\}u_0=0\,.\end{aligned}$$

By virtue of the regularity of $u_0$, from the previous relation we deduce

$$\left(xf(x)+\frac{2-\varepsilon}{4}h(x)\right)\left\{xh'(x)-xf(x)-\frac{2+\varepsilon}{4}h(x)\right\}$$
$$=xh(x)\left\{xf'(x)+\frac{2-\varepsilon}{4}h'(x)-f(x)-2\,x\,\lambda_0(\varepsilon)P_1\right\}=0$$

Some computation arrangements allow to transform this last equality into (4.2.61). □

Consider the expansion of the polynomials $h$ and $f$ according to (4.2.47). Upon the replacement of the polynomials $f(\cdot)$ and $h(\cdot)$ by their expansions, the relation (4.2.61) may be rearranged into

$$\sum_{\nu=0}^{6}\varpi_\nu\,x^\nu=0$$

which naturally implies

$$\varpi_\nu=0,\quad \nu=0,1,2,\ldots,6\;. \tag{4.2.62}$$

After performing these algebraic computations in *Mathematica 6.0*©, the expressions for $\varpi_i$ are the following ones:

$$\begin{aligned}
\varpi_6 &= -f_2^2+\tfrac{1}{16}\left(\varepsilon^2-4\right)h_3^2+\left(2\lambda_0+f_2\right)h_3\\
\varpi_5 &= 2\lambda_0h_2+\tfrac{1}{8}\left(16\lambda_0\,b_1+\left(\varepsilon^2-4\right)h_2\right)h_3+2f_1\left(h_3-f_2\right)\\
\varpi_4 &= -f_1^2+h_2f_1+\tfrac{1}{16}\Big(h_2\left(32\lambda_0\,b_1+\left(\varepsilon^2-4\right)h_2\right)+16f_0\left(3h_3-2f_2\right)\\
&\quad +2h_1\left(16\lambda_0-8f_2+\left(\varepsilon^2-4\right)h_3\right)\Big)\\
\varpi_3 &= \tfrac{1}{8}\Big(16f_0\left(h_2-f_1\right)+h_1\left(16\lambda_0\,b_1+\left(\varepsilon^2-4\right)h_2\right)\\
&\quad +h_0\left(16\lambda_0-16f_2+\left(\varepsilon^2-4\right)h_3\right)\Big)\\
\varpi_2 &= \tfrac{1}{16}\left(-16f_0^2+16h_1f_0+32\lambda_0\,b_1h_0-16f_1h_0+\left(\varepsilon^2-4\right)\left(h_1^2+2h_0h_2\right)\right)\\
\varpi_1 &= \tfrac{1}{8}\left(\varepsilon^2-4\right)h_0h_1\\
\varpi_0 &= \tfrac{1}{16}\left(\varepsilon^2-4\right)h_0^2
\end{aligned}$$

Before entering into further details, it should be pointed out that the condition $\varpi_0=0$ automatically provides $\varpi_1=0$ and it also provides that $\varpi_3=0$ and $\varpi_2=0$ become respectively like

$$16f_0\left(h_2-f_1\right)+h_1\left(16\lambda_0\,b_1+\left(\varepsilon^2-4\right)h_2\right)+16h_0\left(\lambda_0-f_2\right)=0$$
$$-16f_0^2+16h_1f_0+16\big(2\lambda_0\,b_1-f_1\big)h_0+\left(\varepsilon^2-4\right)h_1^2=0$$

As a result the system (4.2.62) may be simplified into the following one:

- $-f_2^2 + \frac{\varepsilon^2-4}{16}\, h_3^2 + (2\lambda_0 + f_2)\, h_3 = 0 \qquad (4.2.63)$
- $2\lambda_0 h_2 + h_3\left(2\lambda_0\, b_1 + \frac{\varepsilon^2-4}{8}\, h_2\right) + 2f_1\,(h_3 - f_2) = 0 \qquad (4.2.64)$
- $f_1(h_2 - f_1) + h_2\Big(2\lambda_0\, b_1 + \frac{\varepsilon^2-4}{16}\, h_2\Big) + f_0\,(3h_3 - 2f_2) + h_1\Big(2\lambda_0 - f_2 - \frac{4-\varepsilon^2}{8}\, h_3\Big) = 0 \qquad (4.2.65)$
- $f_0\,(h_2 - f_1) + h_1\Big(\lambda_0\, b_1 - \frac{4-\varepsilon^2}{16}\, h_2\Big) + h_0\,(\lambda_0 - f_2) = 0 \qquad (4.2.66)$
- $f_0(h_1 - f_0) + \big(2\lambda_0\, b_1 - f_1\big)h_0 - \frac{4-\varepsilon^2}{16}\, h_1^2 = 0 \qquad (4.2.67)$
- $(2-\varepsilon)\, h_0 = 0\ . \qquad (4.2.68)$

In view of the characterisation of $\mathcal{F}_\varepsilon$-classical polynomial sequences, we need to have a more accurate information about the elements that interfere in the differential equation (4.2.37) which has the elements of the $\mathcal{F}_\varepsilon$-classical polynomial sequence $\{P_n\}_{n\in\mathbb{N}}$ as eigenfunctions. The elements in issue are in fact the polynomials $f(\cdot)$, $h(\cdot)$ and also the coefficients $\lambda_0$ and $b_1$, which must satisfy the conditions (4.2.63)-(4.2.68) and also (4.2.57)-(4.2.58). Since the system of equations (4.2.57)-(4.2.58) is more awkward to solve when compared to (4.2.63)-(4.2.68), the key to find these elements lies in (4.2.63)-(4.2.68). Despite of this, the conditions (4.2.57)-(4.2.58) will not be disregarded. The resolution of this problem requires to handle with moderately long computations, which makes the discussion from this point on rather technical.

The outline of the procedure goes as follows. First, we separate two exclusive cases depending on whether $\deg h \leqslant 2$ (Case I) or $\deg h = 3$ (Case II). Based on the assumption taken, the analysis will be drawn up according to the resolution of the nonlinear system given above by (4.2.63)-(4.2.68). After getting more acquainted with the expressions for the polynomials $f(\cdot)$ and $h(\cdot)$, the conditions (4.2.57)-(4.2.58) will be brought into discussion. Notice that the coefficient $h_0$ do not interfere in the conditions (4.2.57)-(4.2.58), but the direct computation of $h(0)$ according to the definition of the polynomial $h(\cdot)$ provided by (4.2.10) allows to overcome this situation. In other words, whenever necessary, we will compute $A_3(0) - (-b_1^{[1]})A_2(0)$ and make the comparison with the obtained expression of $h_0$ from the resolution of (4.2.63)-(4.2.68).

Considering the moderately long computations to be made, during the procedure the symbolic computational language *Mathematica 6.0*© was a useful tool. The commands used were the following ones: `Factor[`*expression*`]`, `Simplify[`*expression*`]`, `Fullsimplify[`*expression*`]`, `Together[`*expression*`]`, `Collect[`*expression*`,{` *variable 1, ..., variable k* `},`*option*`]`.

#### 4.2.3.2 Resolution of the system - analysis of the possible cases

As said before, we consider two different cases, depending on whether $\deg h \leqslant 2$ (Case I) or $\deg h = 3$ (Case II).

### Case I. deg $\mathbf{h}(\cdot) \leqslant \mathbf{2}$

Under this assumption, we have $h_3 = 0$, therefore the conditions (4.2.63)-(4.2.64) successively imply $f_2 = 0$ and $h_2 = 0$ (because $\lambda_0 \neq 0$), providing the condition (4.2.65) to become like $-{f_1}^2 + 2\lambda_0\, h_1 = 0$ yielding

$$h_1 = \frac{{f_1}^2}{2\lambda_0} \tag{4.2.69}$$

Consequently, the condition (4.2.66) may be simplified into the following one

$$2\lambda_0\, h_0 + b_1\, f_1^2 - 2\, f_0\, f_1 = 0$$

permitting to express $h_0$ through

$$h_0 = \frac{2\, f_0\, f_1 - b_1\, f_1^2}{2\lambda_0}\,. \tag{4.2.70}$$

By virtue of the precedent conclusions over the parameters $h_3, h_1, h_0$ and $f_2$, the two conditions (4.2.67)-(4.2.68) may be now written as follows:

$$\Big((2+\varepsilon)f_1^2 + 8f_0\lambda_0 - 8b_1f_1\lambda_0\Big)\Big(-(2-\varepsilon)f_1^2 - 8f_0\lambda_0 + 8b_1f_1\lambda_0\Big) = 0 \tag{4.2.71}$$

$$(2-\varepsilon)\,\frac{2\, f_0 - b_1\, f_1}{2\lambda_0}\; f_1 = 0\,. \tag{4.2.72}$$

On the other hand, following the definition of the polynomial $h(x)$ given in (4.2.10), with $A_{n+1}$ defined in (4.2.11), we obtain

$$h_0 = A_3(0) - b_2^{[1]} A_2(0) \tag{4.2.73}$$

By computing the second member of the previous identity we have

$$A_3(0) - b_2^{[1]} A_2(0) = \frac{b_1 f_1\left((\varepsilon-2)f_1 - 8b_1\lambda_0\right) + 8f_0\left(f_1 + b_1\lambda_0\right)}{2(6+\varepsilon)\lambda_0}$$

therefore, the combination of (4.2.70) with (4.2.73) leads to the conclusion

$$\frac{(f_0 - b_1 f_1)\,(4b_1\lambda_0 - (2+\varepsilon) f_1)}{(\varepsilon+6)\lambda_0} = 0$$

Since $\gamma_1 = \dfrac{f_1 b_1 - f_0}{\lambda_0}$ must be nonzero (otherwise we would be contradicting the regularity of $u_0$), we then have

$$f_1 = \frac{4\,\lambda_0\,b_1}{2+\varepsilon}\ .$$

Consequently, (4.2.71)-(4.2.72) become

$$\Big((2+\varepsilon)f_0 - 2b_1^2\lambda_0\Big)\Big((2+\varepsilon)^2 f_0 - 2(3\varepsilon+2)b_1^2\lambda_0\Big) = 0 \qquad (4.2.74)$$

$$-(2-\varepsilon)\Big((2+\varepsilon)f_0 - 2b_1^2\lambda_0\Big)\,b_1 = 0 \qquad (4.2.75)$$

On the other hand, the conditions (4.2.57)-(4.2.58) are respectively given by

$$-\frac{(2+\varepsilon)^2 + f_0(2+\varepsilon) - 4b_1^2\lambda_0}{2+\varepsilon} = 0 \qquad (4.2.76)$$

$$-\frac{(n+2)(\,\varepsilon\,(4n+\varepsilon+6)+8)\,\big((2+\varepsilon)f_0 - 2b_1^2\lambda_0\big)}{(2+\varepsilon)(2n+\varepsilon+2)(4n+\varepsilon+4)(4n+\varepsilon+8)} = 0\ , \quad n \in \mathbb{N}. \qquad (4.2.77)$$

Insofar as (4.2.77) must hold for the integers $n \in \mathbb{N}$, we conclude that necessarily

$$f_0 = \frac{2\lambda_0\,b_1^2}{2+\varepsilon}$$

and this implies (4.2.74)-(4.2.75) to be identically satisfied and (4.2.76) turns into the condition

$$2\,\lambda_0\,b_1^2 - (2+\varepsilon)^2 = 0$$

compelling $b_1 \neq 0$ because $\varepsilon \neq -2(n+1)$, $n \in \mathbb{N}$,, whence we conclude

$$\lambda_0 = \frac{(2+\varepsilon)^2}{2\,b_1^2}$$

As an outcome, we achieve the expressions for the polynomials $f(\cdot)$ and $h(\cdot)$, which are:

$$\begin{aligned} f(x) &= (2+\varepsilon)\left(\frac{2}{b_1}\,x + 1\right) \\ h(x) &= 4\,x \end{aligned}$$

Hence, according to (4.2.22)-(4.2.24) the semiclassical form $u_0$ is a solution of

$$D(\Phi_i(x)\,u_0) + \Psi_i(x)\,u_0 = 0\ , \qquad i = 1, 2, \qquad (4.2.78)$$

where

$$\Phi_1(x) = x\left(x + \frac{2\,b_1}{2\varepsilon}\right) \quad , \quad \Psi_1(x) = -\frac{((2+\varepsilon)\,x + 2b_1)\,(x(2+\varepsilon) + \varepsilon b_1)}{2(2+\varepsilon)b_1} - \Phi_1'(x)\ ,$$

$$\Phi_2(x) = x^2 \quad , \quad \Psi_2(x) = x\left(-x - \frac{(4+\varepsilon)b_1}{2+\varepsilon} + 2\right) - \Phi_2'(x)\ .$$

The highest common monic factor between $\Phi_1$ and $\Phi_2$ is the polynomial $\Phi(x) = x$. Based on lemma 2.3.2, we realise that $u_0$ is indeed a classical form (or, equivalently, $u_0$ is a semiclassical form of class $s = 0$) since it satisfies the functional equation $D(\Phi u_0) + \Psi u_0 = 0$ with $\Psi(x) = -\frac{2+\varepsilon}{2\,b_1}\,(x + b_1)$. Moreover, the fact that $\Phi(x) = x$ reveals that $u_0$ is a Laguerre form . In order to find the range for the parameter of the Laguerre form $u_0$, we consider the shifted form $\widetilde{u}_0 = h_{a^{-1}}\ u_0$ with $a = -\frac{2+\varepsilon}{2\,b_1}$, which, in accordance with the considerations made in §2.1.1, shares the same properties as those of $u_0$ and it fulfils

$$D(x\,\widetilde{u}_0) + \left(x - \frac{2+\varepsilon}{2}\right)\widetilde{u}_0 = 0\ .$$

Now, recalling the information displayed in Table 2.1 (p. 36), we conclude that $\widetilde{u}_0$, as well as $u_0$, is a Laguerre form of parameter $\varepsilon/2$ and the associated MOPS is a Laguerre polynomial sequence of parameter $\varepsilon/2$.

Besides, from (4.2.8) we have in this case that $u_0$ and $u_0^{[1]}$ are related by

$$4\,x\,u_0^{[1]} = 4\,x\,u_0$$

The division by $x$ on both sides of the previous identity leads, on account of (1.2.6), to $4\,u_0^{[1]} - 4\Big(u_0^{[1]}\Big)_0\,\delta_0 = u_0 - 4(u_0)_0\,\delta_0$, which may be simplified into $u_0^{[1]} = u_0$ . Concomitantly, this last identity discloses the $\mathcal{F}_\varepsilon$-Appell character of $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, since it implies $P_n^{[1]}(\cdot) = P_n(\cdot)$, $n \in \mathbb{N}$, and, on the other hand, it also implies the relation (4.2.6) to become

$$\mathcal{F}_{-\varepsilon} u_0 = \lambda_0 P_1 u_0\ .$$

To sum up, when the assumption $\deg h \leqslant 2$ is taken, necessarily the $\mathcal{F}_\varepsilon$-classical sequence $\{P_n\}_{n\in\mathbb{N}}$ is an orthogonal $\mathcal{F}_\varepsilon$-Appell sequence.

### Case II. deg $\mathbf{h}(\cdot) = \mathbf{3}$

The analysis of this case appears to be a tricky problem and it will be carried out by splitting it into two subcases. The first choice to be considered depends on whether $\varepsilon \neq 2$ (Case II.1) or $\varepsilon = 2$ (Case II.2). The reason behind such partition lies essentially on the nature of the system of equations (4.2.63)- (4.2.68) that ought to be solved. As it will be explained, in what concerns Case II.2, the achieved expressions for the polynomials $f(\cdot)$ and $h(\cdot)$ match those obtained in Case II.1 but with the parameter $\varepsilon$ replaced by $2$. Once the determination of these two polynomials is accomplished in each one of the considered subcases, we will proceed to the assessment of the functional equations (4.2.22)-(4.2.24) fulfilled by $\mathcal{F}_\varepsilon$-classical form $u_0$.

The assumption $\deg h = 3$ entails $h_3 \neq 0$, allowing to obtain from (4.2.63) the identity

$$\lambda_0 = \frac{\Big(4f_2 - (2-\varepsilon)h_3\Big)\Big(4f_2 - (2+\varepsilon)h_3\Big)}{32\; h_3} \tag{4.2.79}$$

Such expression has implications over the expression of $\lambda_n$, for $n \in \mathbb{N}$, and we have from (4.2.51) and (4.2.79)

$$\lambda_n = \frac{\Big(4f_2 + (4n-2-\varepsilon)h_3\Big)\Big(4f_2 + (4n-2+\varepsilon)h_3\Big)}{32 h_3}\,, \qquad n \in \mathbb{N}, \tag{4.2.80}$$

which, because $\lambda_n \neq 0$, presupposes

$$\Big(4f_2 + (4n-2-\varepsilon)h_3\Big)\Big(4f_2 + (4n-2+\varepsilon)h_3 \neq 0, \qquad n \in \mathbb{N}.$$

After the replacement in (4.2.64)-(4.2.68) of $\lambda_0$ by its previous expression (4.2.79), the mentioned conditions become respectively like:

- $$\begin{aligned} & h_3\bigg(\tfrac{\big(4f_2-(2-\varepsilon)h_3\big)\big(4f_2-(2+\varepsilon)h_3\big)}{16\; h_3}\, b_1 + \tfrac{\varepsilon^2-4}{8}\; h_2\bigg) + 2f_1\,(h_3 - f_2) \\ & + \tfrac{\big(4f_2-(2-\varepsilon)h_3\big)\big(4f_2-(2+\varepsilon)h_3\big)}{16\; h_3} h_2 = 0 \end{aligned} \tag{4.2.81}$$
- $$\begin{aligned} & f_1(h_2 - f_1) + h_2\bigg(\tfrac{\big(4f_2-(2-\varepsilon)h_3\big)\big(4f_2-(2+\varepsilon)h_3\big)}{16\; h_3}\, b_1 + \tfrac{\varepsilon^2-4}{16}\; h_2\bigg) \\ & + f_0\,(3h_3 - 2f_2) + h_1\bigg(\tfrac{\big(4f_2-(2-\varepsilon)h_3\big)\big(4f_2-(2+\varepsilon)h_3\big)}{16\; h_3} - f_2 - \tfrac{4-\varepsilon^2}{8}\; h_3\bigg) = 0 \end{aligned} \tag{4.2.82}$$
- $$\begin{aligned} & h_1\bigg(\tfrac{\big(4f_2-(2-\varepsilon)h_3\big)\big(4f_2-(2+\varepsilon)h_3\big)}{32\; h_3}\, b_1 - \tfrac{4-\varepsilon^2}{16}\; h_2\bigg) + f_0\,(h_2 - f_1) \\ & + h_0\bigg(\tfrac{\big(4f_2-(2-\varepsilon)h_3\big)\big(4f_2-(2+\varepsilon)h_3\big)}{32\; h_3} - f_2\bigg) = 0 \end{aligned} \tag{4.2.83}$$
- $$f_0(h_1 - f_0) + \bigg(\tfrac{\big(4f_2-(2-\varepsilon)h_3\big)\big(4f_2-(2+\varepsilon)h_3\big)}{16\; h_3}\, b_1 - f_1\bigg) h_0 - \tfrac{4-\varepsilon^2}{16}\; h_1^2 = 0 \tag{4.2.84}$$
- $$(2-\varepsilon)\, h_0 = 0\;. \tag{4.2.85}$$

The nonzero coefficient $\gamma_1$ can be computed either from the identity $\lambda_0\gamma_1 = 2+\varepsilon$ (associated to the definition of the parameter $\lambda_0$) or also from the relation (4.2.44) through the identity $\gamma_1 = -c_1 - b_1(b_1 - b_2)$ and, considering (4.2.52)-(4.2.54) together with (4.2.80) this last identity becomes

$$\gamma_1 = \frac{32\Big(f_0 + b_1(-f_1 + b_1 f_2)\Big)h_3}{-(4f_2 + (2+\varepsilon)h_3)\;(4f_2 + (2-\varepsilon)h_3)}\;.$$

Notwithstanding this last identity might look misplaced in the text, it is important to notice it here, regarding that $\gamma_1 \neq 0$ will enable us to reject some of the possible solutions of the system (4.2.81)-(4.2.85) that are not valid solutions for the problem in hands.

## Case II.1 : $\varepsilon \neq 2$

Under this assumption, the condition (4.2.85) yields $h_0 = 0$, therefore the conditions (4.2.83)-(4.2.84) are rewritten as follows

$$\begin{aligned}&16\,(h_2 + b_1 h_3)\,f_2^2 - 16 h_3\,(2f_1 + h_2 + b_1 h_3)\,f_2\\&\quad + h_3^2\Big\{\,32 f_1 + \big(\varepsilon^2 - 4\big)\,(h_2 - b_1 h_3)\,\Big\} = 0\end{aligned}\tag{4.2.86}$$

$$\begin{aligned}&16\big(h_1 + b_1 h_2\big) f_2^2 - 16\big(2f_0 + 2h_1 + b_1 h_2\big)h_3 f_2 + h_3\Big\{ - 16 f_1^2 + 16 h_2 f_1\\&\quad + \big(\varepsilon^2 - 4\big)h_2^2 + \big(48 f_0 + \big(\varepsilon^2 - 4\big)h_1\big)h_3 - \big(\varepsilon^2 - 4\big)b_1 h_2 h_3\Big\} = 0\end{aligned}\tag{4.2.87}$$

$$\begin{aligned}&32\ f_0\ h_3\ \ (h_2 - f_1) + h_1\Big\{\Big(4f_2 - (2-\varepsilon)h_3\Big)\Big(4f_2 - (2+\varepsilon)h_3\Big)\,b_1\\&\qquad - 2(4-\varepsilon^2)\ h_2\ h_3\Big\} = 0\end{aligned}\tag{4.2.88}$$

$$\Big(4f_0 - (2-\varepsilon)h_1\Big)\Big(4f_0 - (2+\varepsilon)h_1\Big) = 0 \tag{4.2.89}$$

From (4.2.89) we deduce that either $f_0 = \frac{2+\varepsilon}{4}\,h_1$ or $f_0 = \frac{2-\varepsilon}{4}\,h_1$, which implies the condition (4.2.88) to be transformed into

$$h_1\Big\{h_2(2+\varepsilon)^2 - 4f_1(2+\varepsilon) + 16\,b_1\lambda_0\Big\} = 0 \tag{4.2.90}$$

when $f_0 = \frac{2+\varepsilon}{4}\,h_1$, or into

$$h_1\Big\{h_2(2-\varepsilon)^2 - 4f_1(2-\varepsilon) + 16\,b_1\lambda_0\Big\} = 0 \tag{4.2.91}$$

if $f_0 = \frac{2-\varepsilon}{4}\,h_1$. At this point, we shall consider the two exclusive cases depending on whether $h_1 \neq 0$ (Case II.1.1) or $h_1 = 0$ (Case II.1.2).

## Case II.1.1 : $\mathbf{h_1 \neq 0}$

Under this assumption there are two possibilities already mentioned: $f_0 = \frac{2+\varepsilon}{4}\, h_1$ or $f_0 = \frac{2-\varepsilon}{4}\, h_1$, which we will be referring to as Subcase A and Subcase B.

**Subcase A:** $f_0 = \dfrac{2+\varepsilon}{4}\, h_1$

Under the assumption, from (4.2.90) we deduce

$$f_1 = \frac{2h_2h_3(2+\varepsilon)^2 + b_1\left(4f_2-(2-\varepsilon)h_3\right)\left(4f_2-(2+\varepsilon)h_3\right)}{8(2+\varepsilon)h_3}\,. \tag{4.2.92}$$

The relations (4.2.86)-(4.2.87) with $f_1$ and $f_0$ replaced by the corresponding achieved expressions may be respectively written like

$$\begin{aligned}\Big(4f_2-(2+\varepsilon)h_3\Big)\Big(4f_2-(6+\varepsilon)h_3\Big)\Big(&-(2+\varepsilon)h_2\\ &+b_1\left(4f_2-(2-\varepsilon)h_3\right)\Big)=0\end{aligned} \tag{4.2.93}$$

$$\begin{aligned}\left(4f_2-(2+\varepsilon)h_3\right)\ \Big\{&4h_1h_3\left((10+\varepsilon)h_3-4f_2\right)(2+\varepsilon)^2\\ &+b_1^2\left(4f_2-(2-\varepsilon)h_3\right)^2\left(4f_2-(2+\varepsilon)h_3\right)\\ &-8b_1h_2h_3\left(4f_2-(2-\varepsilon)h_3\right)(2+\varepsilon)\Big\}=0\end{aligned} \tag{4.2.94}$$

Since $\lambda_0 \neq 0$, we necessarily have $4f_2-(2+\varepsilon)h_3 \neq 0$. So the condition (4.2.93) provides that either $4f_2-(6+\varepsilon)h_3 = 0$ or $-(2+\varepsilon)h_2 + b_1\left(4f_2-(2-\varepsilon)h_3\right) = 0$.

Actually, we cannot afford to have $4f_2-(6+\varepsilon)h_3 = 0$, otherwise the condition (4.2.94) would be reduced to $-h_3\left(h_1+b_1\left(b_1h_3-h_2\right)\right) = 0$ and, because $h_3 \neq 0$, it would imply $h_1 = -b_1\left(b_1h_3-h_2\right)$, providing $\gamma_1 = 0$, which denies the regular orthogonality of $\{P_n\}_{n\in\mathbb{N}}$. Consequently, we surely have

$$4f_2-(6+\varepsilon)h_3 \neq 0$$

and from the relation (4.2.93) we shall simply read

$$h_2 = \frac{b_1\left(4f_2-(2-\varepsilon)h_3\right)}{2+\varepsilon} \tag{4.2.95}$$

which enables to transform (4.2.94) into

$$\Big((2+\varepsilon)h_3-4f_2\Big)\Big((10+\varepsilon)h_3-4f_2\Big)\Big(b_1^2\left(4f_2-(2-\varepsilon)h_3\right)^2-4(2+\varepsilon)^2h_1h_3\Big)=0$$

Since $\lambda_0 \neq 0$, then, recalling (4.2.79), we either have

$$(10+\varepsilon)h_3-4f_2=0 \qquad \textbf{or} \qquad b_1^2\left(4f_2-(2-\varepsilon)h_3\right)^2-4(2+\varepsilon)^2h_1h_3=0\,. \tag{4.2.96}$$

Meanwhile, upon the precedent conclusions, the condition (4.2.58) with the particular choice of $n=0$ gives place to

$$\frac{(6+\varepsilon)\left((2+\varepsilon)h_3-4f_2\right)\left(b_1^2\left(4f_2-(2-\varepsilon)h_3\right)^2-4(2+\varepsilon)^2h_1h_3\right)}{4(2+\varepsilon)^2h_3\left((-(2-\varepsilon)\varepsilon-32)h_3-4(8+\varepsilon)f_2\right)}=0\ .$$

This last equality, associated with the fact that $\lambda_0\neq0$, permits to conclude that

$$h_1=\frac{b_1^2\left(4f_2-(2-\varepsilon)h_3\right)^2}{4(2+\varepsilon)^2\,h_3} \tag{4.2.97}$$

but with

$$f_2\neq\frac{(-(2-\varepsilon)\varepsilon-32)h_3}{4(8+\varepsilon)} \tag{4.2.98}$$

because $\lambda_0\neq0$. Upon such conclusions, the condition (4.2.96) is identically satisfied, and (4.2.49)-(4.2.54) may be written like:

$$\frac{(4f_2+(8n+6+\varepsilon)h_3)\left(4(4n+6+\varepsilon)f_2+\left(16n^2+4(6+\varepsilon)n-\varepsilon^2+4\right)h_3\right)}{32h_3}\,b_{n+2}$$
$$=\frac{(n+2)(2n+\varepsilon+4)b_1(4f_2-(2-\varepsilon)h_3)\left(4(4n+6+\varepsilon)f_2+\left(16n^2+4(6+\varepsilon)n-\varepsilon^2+4\right)h_3\right)}{32(2+\varepsilon)h_3}\ ,\quad n\in\mathbb{N},$$

and

$$\frac{(4f_2+(8n+2+\varepsilon)h_3)\left(4(4n+\varepsilon+4)f_2+\left(16n^2+4(2+\varepsilon)n+(2-\varepsilon)(4+\varepsilon)\right)h_3\right)}{16h_3}\,c_{n+1}$$
$$=\frac{(n+1)(n+2)(2n+2+\varepsilon)(2n+\varepsilon+4)\,b_1^2\left(4f_2-(2-\varepsilon)h_3\right)^2\left(4(4n+\varepsilon+4)f_2+\left(16n^2+4(2+\varepsilon)n+(2-\varepsilon)(4+\varepsilon)\right)h_3\right)}{32(2+\varepsilon)^2h_3(4f_2+(8n+6+\varepsilon)h_3)}$$

for $n\in\mathbb{N}$. Assuming that

$$\left(4(4n+\varepsilon+4)f_2+\left(16n^2+4(2+\varepsilon)n+(2-\varepsilon)(4+\varepsilon)\right)h_3\right)\neq0\ ,\qquad n\in\mathbb{N},$$

we deduce

$$\begin{aligned} b_{n+2}&=\frac{b_1\,(n+2)(2n+\varepsilon+4)\,(4f_2-(2-\varepsilon)h_3)}{(2+\varepsilon)\,(4f_2+(8n+6+\varepsilon)h_3)}\ ,\quad n\in\mathbb{N},\\ c_{n+1}&=\frac{b_1^2\,(n+1)(n+2)(2n+2+\varepsilon)(2n+\varepsilon+4)\,(4f_2-(2-\varepsilon)h_3)^2}{2(2+\varepsilon)^2\,(4f_2+(8n+\varepsilon+2)h_3)\,(4f_2+(8n+6+\varepsilon)h_3)}\ , \end{aligned} \tag{4.2.99}$$

for any $n\in\mathbb{N}$, which allows to say that (4.2.58) is identically satisfied for all the integers $n\in\mathbb{N}$. Consequently, from (4.2.57) we conclude

$$\begin{aligned}&b_1^2\,(4f_2-(2-\varepsilon)h_3)\,(4f_2-(2+\varepsilon)h_3)\,(4f_2-(6+\varepsilon)h_3)\\&-16(2+\varepsilon)^2h_3\,(4f_2+(2+\varepsilon)h_3)=0\end{aligned}$$

permiting to express $b_1$ in terms of $f_2$, $h_3$ and $\varepsilon$:

$${b_1}^2=\frac{16(2+\varepsilon)^2\ (4f_2+(2+\varepsilon)h_3)\ h_3}{(4f_2-(2-\varepsilon)h_3)\,(4f_2-(2+\varepsilon)h_3)\,(4f_2-(6+\varepsilon)h_3)} \tag{4.2.100}$$

obliging, in particular, $b_1$ to be nonzero, otherwise we would be contradicting the condition $\lambda_1 \neq 0$ or the assumption $h_3 \neq 0$ or the constraints for the parameter $\varepsilon \neq -2n,\ n \geqslant 1$. To avoid the use of long expressions we will not replace in the expression for $b_n$ or $c_n$ the obtained expression for $b_1$. Untill the end of the study of this subcase we will use $b_1$, despite knowing how to express it in terms of $f_2, h_3$ and $\varepsilon$. So far, the three parameters $f_2, h_3$ and $\varepsilon$ remain undetermined. The foregoing conclusions allow to extract

$$\beta_n = \frac{-b_1\,(4f_2-(2-\varepsilon)h_3)}{(2+\varepsilon)\,(4f_2+(8n+\varepsilon-10)h_3)\,(4\,f_2+(8n+\varepsilon-2)h_3)} \times \big(4(4n+2+\varepsilon)f_2+\big(16n^2-4(6-\varepsilon)n-(10-\varepsilon)(2+\varepsilon)\big)\,h_3\big)\ ,\ n\in\mathbb{N},$$

$$\gamma_{n+1} = \frac{2(n+1)(2n+2+\varepsilon)b_1^2\,(4f_2+(4n-\varepsilon-6)h_3)}{(2+\varepsilon)^2\,(4f_2+(8n+\varepsilon-6)h_3)\,(4f_2+(8n+\varepsilon-2)h_3)^{\,2}} \times \frac{(4f_2-(2-\varepsilon)h_3)^{\,2}\,(4f_2+(4n+\varepsilon-6)h_3)}{(4f_2+(8n+2+\varepsilon)h_3)}\ ,\quad n\in\mathbb{N}.$$

To accomplish the purpose of finding the regular form $u_0$ fulfilling (4.2.22)-(4.2.24), we need the expressions for the polynomials $f$ and $h$, which happen to be

$$f(x) = \left(f_2\,x+\frac{b_1\big(4f_2-(2-\varepsilon)h_3\big)}{8}\right)\left(x+\frac{b_1\big(4f_2-(2-\varepsilon)h_3\big)}{2(2+\varepsilon)h_3}\right) \tag{4.2.101}$$

and

$$h(x) = h_3\,x\left(x+\frac{b_1\big(4f_2-(2-\varepsilon)h_3\big)}{2(2+\varepsilon)h_3}\right)^2 \tag{4.2.102}$$

with $b_1$ satisfying (4.2.100).

In the upcoming cases, the aforementioned expressions for the polynomials $f(\cdot)$ and $h(\cdot)$ will be retrieved.

**Subcase B:** $f_0 = \dfrac{2-\varepsilon}{4}\,h_1$

Under this assumption, the system of conditions (4.2.86)-(4.2.89), is reduced to the system obtained under the hypothesis made in "Subcase A" (4.2.92)-(4.2.96), but with $(-\varepsilon)$ instead of $\varepsilon$. Resuming, either we have

$$f_2 = \frac{10-\varepsilon}{4}\,h_3 \qquad \text{or} \qquad h_1 = \frac{b_1^2\,\big((2+\varepsilon)h_3-4f_2\big)^2}{4(2-\varepsilon)^2h_3}\ . \tag{4.2.103}$$

Let us analyse separately each one of the only two possible situations in this case mentioned in (4.2.103).

**Situation B.1:** $f_2 = \dfrac{10-\varepsilon}{4}\, h_3$

In this case we have

$$\lambda_n = \frac{1}{4}(n+2)(2n-\varepsilon+4)h_3 \qquad n \in \mathbb{N},$$

In particular this implies that $\varepsilon \neq 2(n+1)$, $n \in \mathbb{N}$ because $\lambda_n$ must be nonzero for all the integers $n \in \mathbb{N}$. On the other hand, from relation (4.2.57) we have

$$\frac{-(4-\varepsilon)(2-\varepsilon)^2\, h_1 - 6(6-\varepsilon)(2+\varepsilon)(2-\varepsilon) + (4-\varepsilon)(-(8-\varepsilon)\,\varepsilon+28)b_1^2 h_3}{6(6-\varepsilon)(2-\varepsilon)} = 0$$

providing

$$h_1 = \frac{-6(6-\varepsilon)(2+\varepsilon)}{(4-\varepsilon)(2-\varepsilon)} + \frac{(-(8-\varepsilon)\,\varepsilon+28)b_1^2 h_3}{(2-\varepsilon)^2}$$

Therefore, the relation (4.2.58) with $n=0$ becomes

$$\frac{-144(6-\varepsilon)(2+\varepsilon)}{(4-\varepsilon)\,(2-\varepsilon)\big(-(2-\varepsilon)\,\varepsilon-56\big)} = \frac{72\big(-(6-\varepsilon)\,\varepsilon\,(2-\varepsilon)^2+128\big)b_1^2 h_3}{(2-\varepsilon)^2\big(-(2-\varepsilon)\,\varepsilon-56\big)\big(-(2-\varepsilon)\,\varepsilon-32\big)}$$

while (4.2.58) with $n=1$ brings

$$\frac{432(6-\varepsilon)(2+\varepsilon)^2\Big((2-\varepsilon)\varepsilon+64\Big)}{(4-\varepsilon)\,(2-\varepsilon)(4+\varepsilon)\Big((2-\varepsilon)\varepsilon+104\Big)\Big((2-\varepsilon)\varepsilon+56\Big)}$$
$$= \frac{72(2+\varepsilon)\Bigg(-(2-\varepsilon)\,\varepsilon\,\Big(\varepsilon\,(\varepsilon(3(\varepsilon-10)\varepsilon-68)+1144)-800\Big)-55296\Bigg)b_1^2 h_3}{(2-\varepsilon)^2\,(4+\varepsilon)\Big(-(2-\varepsilon)\,\varepsilon-104\Big)\Big(-(2-\varepsilon)\,\varepsilon-72\Big)\Big(-(2-\varepsilon)\,\varepsilon-56\Big)}$$

The first relation enables the identity

$$b_1^2 = \frac{2(2-\varepsilon)(6-\varepsilon)(2+\varepsilon)\,(\varepsilon^2-2\varepsilon-32)}{(\varepsilon-4)\,(\varepsilon^4-10\varepsilon^3+28\varepsilon^2-24\varepsilon+128)\,h_3}$$

which replaced in the second condition provides

$$-\frac{23040\,\varepsilon\,(4-\varepsilon)(6-\varepsilon)(2+\varepsilon)^2}{(\varepsilon+4)((\varepsilon-2)\varepsilon-104)((\varepsilon-2)\varepsilon-72)\,((\varepsilon-6)\varepsilon(\varepsilon-2)^2+128)} = 0$$

whence we deduce $\varepsilon = 0$. The case under analysis corresponds to the particular choice of Case A when $\varepsilon = 0$ and it was previously analysed.

**Situation B.2:** $h_1 = \dfrac{b_1^2\,\Big((2+\varepsilon)h_3 - 4f_2\Big)^2}{4(2-\varepsilon)^2 h_3}$

Under the assumption from relation (4.2.58) with $n=0$ we get

$$-\frac{3\,\varepsilon\ (4f_2-(2-\varepsilon)h_3)\,((2+\varepsilon)h_3-4f_2)^{\,2}\,(4f_2+(2+\varepsilon)h_3)\,b_1^2}{-4(2-\varepsilon)h_3\,((\varepsilon^2-4)\,h_3-4(6+\varepsilon)f_2)\,((\varepsilon^2-2\varepsilon-32)\,h_3-4(8+\varepsilon)f_2)}=0$$

Since $\lambda_0\lambda_1\neq 0$, from the previous identity we deduce that $\varepsilon=0$. Again, this subcase results in a specialisation of Subcase A.

### Case II.1.2 : $h_1=0$

We begin by analysing the consequences of the assumption over the conditions (4.2.86)-(4.2.88). In particular, from (4.2.89) we get

$$f_0=0$$

providing (4.2.86)-(4.2.87) to become respectively like

$$\begin{aligned}16\,(h_2+b_1h_3)\,f_2^2-16h_3\,(2f_1+h_2+b_1h_3)\,f_2+h_3^2\left(32f_1+\left(\varepsilon^2-4\right)(h_2-b_1h_3)\right)=0\\ \left(-16f_1^2+16h_2f_1+\left(\varepsilon^2-4\right)h_2^2\right)h_3+b_1\ h_2\,\left(4f_2-(2-\varepsilon)h_3\right)\left(4f_2-(2+\varepsilon)h_3\right)=0\end{aligned}$$

that is

$$\begin{aligned}&16h_2f_2^2-16\,(2f_1+h_2)\,h_3f_2+\left(32f_1+\left(\varepsilon^2-4\right)h_2\right)h_3^2\\ &+h_3\,b_1\ (4f_2-(2-\varepsilon)h_3)\,(4f_2-(2+\varepsilon)h_3)=0\end{aligned}\tag{4.2.104}$$

$$\begin{aligned}&\left(-16f_1^2+16h_2f_1+\left(\varepsilon^2-4\right)h_2^2\right)h_3\\ &+b_1\ h_2\,\left(4f_2-(2-\varepsilon)h_3\right)\left(4f_2-(2+\varepsilon)h_3\right)=0\end{aligned}\tag{4.2.105}$$

Once more, insofar as $\lambda_0\neq 0$, then, recalling (4.2.79), we obtain, as an outcome of (4.2.104),

$$b_1=\frac{16h_2f_2^2-16\,(2f_1+h_2)\,h_3f_2+\left(32f_1+\left(\varepsilon^2-4\right)h_2\right)h_3^2}{h_3\,b_1\ (4f_2-(2-\varepsilon)h_3)\,(4f_2-(2+\varepsilon)h_3)}\tag{4.2.106}$$

which permits to simplify (4.2.105) into

$$-\frac{(f_2h_2-f_1h_3)\,(f_2h_2-(f_1+h_2)\,h_3)}{h_3^2}=0\ ,$$

whence, necessarily, we have

$$f_1=\frac{f_2\,h_2}{h_3}-\xi\,h_2\qquad\text{with}\qquad \xi=0\quad\text{or}\quad\xi=1\ .\tag{4.2.107}$$

Following (4.2.49) for $n=2$, we get

$$\frac{(4f_2+(6+\varepsilon)h_3)\left(4(6+\varepsilon)f_2+\left(4-\varepsilon^2\right)h_3\right)}{32h_3}\,b_2 = \frac{(4+\varepsilon)h_2\,(4f_2+(2-\varepsilon)h_3)\,(4f_2+(2+\varepsilon)h_3)}{16h_3^2}$$

when $\xi=0$ and

$$\frac{(4f_2+(6+\varepsilon)h_3)\left(4(6+\varepsilon)f_2+\left(4-\varepsilon^2\right)h_3\right)}{32h_3}\,b_2 = \frac{(4+\varepsilon)h_2\,(4f_2-(2-\varepsilon)h_3)\,(4f_2-(2+\varepsilon)h_3)}{16h_3^2}$$

if $\xi=1$. Since $\lambda_0\lambda_1\lambda_2\neq 0$, we have

$$\Big(4f_2-(2-\varepsilon)h_3\Big)\Big(4f_2-(2+\varepsilon)h_3\Big)\Big(4f_2+(2-\varepsilon)h_3\Big)\Big(4f_2+(2+\varepsilon)h_3\Big)\Big(4f_2+(6+\varepsilon)h_3\Big)\neq 0$$

Hence $4(6+\varepsilon)f_2+\left(4-\varepsilon^2\right)h_3=0$ if and only if $h_2=0$ (no matter the possible value for $\xi$), which in turn would imply $b_1=0$ and consequently we would be contradicting the regularity of $u_0$, because that would imply $\gamma_1=0$. Thus, necessarily $h_2\Big(4(6+\varepsilon)f_2+\left(4-\varepsilon^2\right)h_3\Big)\neq 0$, and we deduce

$$b_2=\frac{2\,(4+\varepsilon)h_2\,(4f_2+(2-\varepsilon)h_3)\,(4f_2+(2+\varepsilon)h_3)}{h_3\;(4f_2+(6+\varepsilon)h_3)\,(4(6+\varepsilon)f_2+(4-\varepsilon^2)\,h_3)}\quad \text{if}\quad \xi=0$$

or

$$b_2=\frac{(4+\varepsilon)h_2\,(4f_2-(2-\varepsilon)h_3)\,(4f_2-(2+\varepsilon)h_3)}{h_3\;(4f_2+(6+\varepsilon)h_3)\,(4(6+\varepsilon)f_2+(4-\varepsilon^2)\,h_3)}\quad \text{if}\quad \xi=1$$

For any of the possible values of $\xi$, we have

$$\begin{aligned} c_1 \;=\; & \frac{h_2^2\,(2+\varepsilon)\big(-16f_2^2+16(2\xi-1)h_3f_2-(4-\varepsilon^2)h_3^2\big)}{h_3^2\big(4f_2+(2-\varepsilon)h_3\big)\big(4f_2+(2+\varepsilon)h_3\big)} \\ & \cdot\frac{\big(-16f_2^2+16(2\xi+1)h_3f_2+\left(\varepsilon^2-32\xi-4\right)h_3^2\big)}{\big(4f_2+(6+\varepsilon)h_3\big)\big(4(6+\varepsilon)f_2+(4-\varepsilon^2)h_3\big)} \end{aligned}$$

Therefore, according to (4.2.44) it follows

$$\begin{aligned} \gamma_1 \;=\; & \frac{32\,\xi\,h_2^2\big(16f_2^2-16h_3f_2+\left(\varepsilon^2-4\right)h_3^2\big)}{\big(4f_2-(2+\varepsilon)h_3\big)^2\big(4f_2+(2+\varepsilon)h_3\big)} \\ & \cdot\frac{\big(-16f_2^2+16(2\xi+1)h_3f_2+\left(\varepsilon^2-32\xi-4\right)h_3^2\big)}{\big(4f_2-(2+\varepsilon)h_3\big)^2\big(4f_2+(2+\varepsilon)h_3\big)} \end{aligned}$$

The fact that $\gamma_1\neq 0$, requires $\xi\neq 0$, whence $\xi=1$ and according to (4.2.107)

$$f_1=\frac{f_2\,h_2}{h_3}-\,h_2$$

whence

$$\gamma_1 = \frac{-32\, h_2^2\, (4f_2 + (\varepsilon-6)h_3)\,(4f_2-(6+\varepsilon)h_3)\,\left(16f_2^2 - 16h_3f_2 + \left(\varepsilon^2-4\right)h_3^2\right)}{-\,(4f_2-(\varepsilon-2)h_3)\,(4f_2-(2-\varepsilon)h_3)^{\,2}\,((2+\varepsilon)h_3-4f_2)^{\,2}\,(4f_2+(2+\varepsilon)h_3)}$$

Instead of proceeding to compute the conditions (4.2.57)-(4.2.58), we compute $h(0)$ from its definition (4.2.10). More precisely, we compute $h(0) = A_3(0) - b_2^{[1]}A_2(0)$ (with $A_{n+1}$, $n \in \mathbb{N}$, defined in (4.2.11)). Under the assumptions, it was expected to have $h(0) = h_0 = 0$; notwithstanding we have

$$h(0) = A_3(0) - b_2^{[1]}A_2(0) = \frac{4h_2^3\,(4f_2+\varepsilon h_3-6h_3)\,(-4f_2+\varepsilon h_3+6h_3)}{h_3^2\,(-4f_2+\varepsilon h_3-2h_3)\,(h_3\varepsilon^2-4f_2\varepsilon-24f_2-4h_3)}$$

which cannot be zero, otherwise we would be contradicting $\gamma_1 \neq 0$. In brief: the assumption $h_1 = 0$ with $h_3 \neq 0$ and $\varepsilon \neq 2$ contradicts the regularity of $u_0$, ergo $h_1 \neq 0$ whenever $h_3 \neq 0$ and $\varepsilon \neq 2$.

## Case II.2 : $\varepsilon = 2$

In this case the conditions (4.2.81)-(4.2.84) become

$$\frac{(f_2-h_3)\,(f_2\,(h_2+b_1h_3)-2f_1h_3)}{h_3} = 0 \tag{4.2.108}$$

$$\frac{(h_1+b_1h_2)\,f_2^2-(2f_0+2h_1+b_1h_2)\,h_3f_2+h_3\left(-f_1^2+h_2f_1+3f_0h_3\right)}{h_3} = 0 \tag{4.2.109}$$

$$\frac{(h_0+b_1h_1)\,f_2^2-(3h_0+b_1h_1)\,h_3f_2+2f_0\,(h_2-f_1)\,h_3}{h_3} = 0 \tag{4.2.110}$$

$$-\,f_0^2+h_1f_0+\frac{h_0\,(b_1f_2\,(f_2-h_3)-f_1h_3)}{h_3} = 0 \tag{4.2.111}$$

and (4.2.80) is simplified into

$$\lambda_n = \frac{\big(f_2+(n-1)h_3\big)\big(f_2+n\,h_3\big)}{2\,h_3}\ , \quad n \in \mathbb{N}.$$

Considering that $\lambda_0 \neq 0$, the condition (4.2.108) yields

$$h_2 = \left(-b_1 + \frac{2\,f_1}{f_2}\right)h_3\ .$$

By replacing in (4.2.109)-(4.2.111) $h_2$ by the foregoing expression we respectively obtain:

$$\begin{aligned}&\left(\frac{2h_3}{f_2}-1\right)f_1^2+b_1\left(2f_2-3h_3\right)f_1+f_0\left(3h_3-2\,f_2\right)\\&\qquad+f_2\left(\left(h_3-f_2\right)b_1^2+h_1\left(\frac{f_2}{h_3}-2\right)\right)=0\end{aligned}\tag{4.2.112}$$

$$f_2\left(-3h_0-b_1h_1+\frac{f_2\left(h_0+b_1h_1\right)}{h_3}\right)+f_0\left(f_1\left(\frac{4h_3}{f_2}-2\right)-2b_1h_3\right)=0\tag{4.2.113}$$

$$f_0\left(h_1-f_0\right)+h_0\left(\frac{b_1f_2\left(f_2-h_3\right)}{h_3}-f_1\right)=0\tag{4.2.114}$$

The relation (4.2.112) permit to obtain an expression for $h_1$ in terms of the remaining parameters unless we have $f_2=2\,h_3$. However, if $f_2=2\,h_3$ the relations (4.2.112)-(4.2.114) would become respectively like

$$\begin{aligned}&-h_3\left(f_0+b_1\left(2b_1h_3-f_1\right)\right)=0\\&-2\left(h_0+b_1\left(f_0-h_1\right)\right)h_3=0\\&-f_0^2+h_1f_0-h_0\left(f_1-2b_1h_3\right)=0\end{aligned}$$

The first condition would then provide $f_0=-b_1\left(2b_1h_3-f_1\right)$ contradicting $\gamma_1\neq 0$. Therefore necessarily $f_2\neq 2\,h_3$, which enables to express from (4.2.112) $h_1$ in terms of the other parameters:

$$h_1=\frac{h_3\left(\left(f_2-2h_3\right)f_1^2+b_1f_2\left(3h_3-2f_2\right)f_1+f_2\left(f_2\left(f_2-h_3\right)b_1^2+f_0\left(2f_2-3h_3\right)\right)\right)}{f_2^2\left(f_2-2h_3\right)}$$

By virtue of this equality, it is possible to rewrite (4.2.113)

$$\begin{aligned}&{f_2}^2\ \left(f_2-2h_3\right)\left(\frac{f_2}{h_3}-3\right)\ h_0+b_1\left(f_1-b_1f_2\right)\left(f_2-h_3\right)\Big\{f_1\left(f_2-2h_3\right)\\&+b_1f_2\left(h_3-f_2\right)\Big\}+f_0\Big(b_1f_2\left(2f_2^2-7h_3f_2+7h_3^2\right)-2f_1\left(f_2-2h_3\right)^2\Big)=0\end{aligned}\tag{4.2.115}$$

while, the condition (4.2.114) becomes

$$\begin{aligned}&h_0\left(\frac{b_1f_2\left(f_2-h_3\right)}{h_3}-f_1\right)\\&+\frac{f_0\left(\left(f_1-b_1f_2\right)h_3\left(f_1\left(f_2-2h_3\right)+b_1f_2\left(h_3-f_2\right)\right)-f_0f_2\left(f_2-3h_3\right)\left(f_2-h_3\right)\right)}{f_2^2\left(f_2-2h_3\right)}=0\end{aligned}$$

that is

$$\begin{aligned}&f_2^2\left(f_2-2h_3\right)\ \left(\frac{b_1f_2\left(f_2-h_3\right)}{h_3}-f_1\right)\ h_0+f_0\Big\{\left(f_1-b_1f_2\right)h_3\Big(f_1\left(f_2-2h_3\right)\\&+b_1f_2\left(h_3-f_2\right)\Big)-f_0f_2\left(f_2-3h_3\right)\left(f_2-h_3\right)\Big\}=0\end{aligned}\tag{4.2.116}$$

So far we have no additional condition over the coefficient $h_0$, not even the conditions (4.2.57)-(4.2.58) can provide such relation. However, this situation may be overcome by computing $h(0) = h_0$ according to

$$\begin{aligned} h_0 &= h(0) = A_3(0) - b_2^{[1]} A_2(0) \\ &= \frac{b_1\left(f_0 + b_1\left(b_1 f_2 - f_1\right)\right)}{4} + \frac{b_1 f_2\left(f_0 + b_1\left(b_1 f_2 - f_1\right)\right)}{4(f_2 - 2h_3)} + \frac{f_0\left(f_1 - b_1 f_2\right) h_3}{f_2^2} \end{aligned} \tag{4.2.117}$$

where $A_{n+1}$, $n \in \mathbb{N}$, is defined in (4.2.11). In (4.2.115) we proceed to the replacement of $h_0$ by the achieved expression and this leads to

$$\frac{\Big(f_0 + b_1\left(b_1 f_2 - f_1\right)\Big)\Big(f_2 - h_3\Big)\Big(b_1 f_2\left(f_2 + h_3\right) - 2 f_1 h_3\Big)}{2 f_2 h_3} = 0$$

yielding

$$f_1 = \frac{b_1\, f_2\, \left(f_2 + h_3\right)}{2\, h_3}\ . \tag{4.2.118}$$

because $\lambda_0 \neq 0$ and $\gamma_1 = -\frac{2\left(f_0 + b_1\left(b_1 f_2 - f_1\right)\right)}{f_2\left(f_2 + h_3\right)} \neq 0$. Consequently, (4.2.117) may be simplified into

$$h_0 = -\frac{b_1\left(f_2 - h_3\right)^2\left(b_1^2 f_2^2 - 4 f_0 h_3\right)}{4 f_2\left(f_2 - 2h_3\right) h_3}\ ,$$

while (4.2.116) may be converted into

$$-\frac{\left(f_2 - 3h_3\right)\left(f_2 - h_3\right)\left(b_1^2 f_2^2 - 4 f_0 h_3\right)\left(b_1^2 f_2\left(f_2 - h_3\right) - 2 f_0 h_3\right)}{8 f_2\left(f_2 - 2h_3\right) h_3^2} = 0\ ,.$$

Inherent to the constraints $\lambda_0 \neq 0$ and $\gamma_1 = \frac{b_1^2 f_2(f_2 - h_3) - 2 f_0 h_3}{f_2(f_2 + h_3)} \neq 0$, this last condition implies that either

$$f_2 = 3h_3 \qquad \text{or} \qquad f_0 = \frac{b_1^2 f_2^2}{4h_3}\ . \tag{4.2.119}$$

On the other hand the relation (4.2.57) becomes

$$\frac{f_0\left(h_3 - f_2\right)}{f_2 + h_3} + \frac{b_1^2 f_2^3 - 2b_1^2 h_3 f_2^2 + b_1^2 h_3^2 f_2 - 8 h_3 f_2 - 8 h_3^2}{2 h_3\left(f_2 + h_3\right)} = 0 \tag{4.2.120}$$

while (4.2.58) with $n = 0$ may be simplified into

$$\frac{\left(b_1^2 f_2^2 - 4 f_0 h_3\right)\left(f_2^2 - h_3^2\right)}{4\left(f_2 - 2h_3\right) h_3\left(5 f_2 + 4 h_3\right)} = 0 \tag{4.2.121}$$

with the necessary restriction

$$5 f_2 + 4 h_3 \neq 0\ .$$

From (4.2.119)-(4.2.121) and on account of the fact $\lambda_0, \lambda_1 \neq 0$, we conclude

$$f_0 = \frac{b_1^2 f_2^2}{4h_3}$$

which implies (4.2.120) to become

$$\frac{b_1^2 f_2 \left(f_2 - 2h_3\right)\left(f_2 - h_3\right) - 16h_3\left(f_2 + h_3\right)}{4h_3\left(f_2 + h_3\right)} = 0$$

thereby

$$b_1^2 = \frac{16h_3\left(f_2 + h_3\right)}{f_2\left(f_2 - 2h_3\right)\left(f_2 - h_3\right)} \tag{4.2.122}$$

After the conclusions, we have

$$\begin{aligned} b_{n+1} &= \frac{(n+1)(n+2)b_1 f_2}{2\left(f_2 + 2nh_3\right)}, \quad n \in \mathbb{N}, \\ c_{n+1} &= \frac{(n+1)(n+2)^2(n+3)b_1^2 f_2^2}{8\left(f_2 + 2(n+1)h_3\right)\left(f_2 + (2n+1)h_3\right)} \end{aligned}$$

where $b_1$ is given by (4.2.122). As a consequence, the relation (4.2.58) is identically satisfied for all the integers $n \in \mathbb{N}$.

As a result, the polynomials $f(x)$ and $h(x)$ are given by

$$f(x) = f_2\left(x + \frac{b_1}{2}\right)\left(x + \frac{b_1 f_2}{2\,h_3}\right) \qquad \text{and} \qquad h(x) = h_3\, x\left(x + \frac{b_1 f_2}{2\,h_3}\right)^2$$

which indeed correspond to the corresponding expressions obtained in Case II.1.1.A but with $\varepsilon = 2$ given by (4.2.101)-(4.2.102).

## Conclusions of Case II

The outcome of having supposed $\deg h = 3$, consists in the fact that the polynomials $f$ and $h$ are given by (4.2.101)-(4.2.102) where $b_1$ satisfies the condition (4.2.100) and the parameter $\varepsilon$ do not have any restrictions to its range (*i.e.* , $\varepsilon \in \mathbb{C}$ with $\varepsilon \neq -2\,n$, for any $n \in \mathbb{N}^*$). Consequently, in accordance with (4.2.22)-(4.2.24), the regular form $u_0$ satisfies

$$D\Big(\Phi_i(x)u_0\Big) + \Psi_i(x)u_0 = 0\,, \quad i = 1, 2, 3,$$

with

$$\begin{aligned} \Phi_1(x) &= x(x - r_1)(x - R) \\ \Psi_1(x) &= -\frac{(x - r_1)}{4h_3}\left(\big(4f_2 - (10 - \varepsilon)h_3\big)\, x + \frac{\varepsilon\, b_1(4f_2 - (2 - \varepsilon)h_3)}{2 + \varepsilon}\right) - \Phi_1'(x)\,, \end{aligned}$$

and

$$\Phi_2(x) = x^2\ (x-R)^2$$
$$\Psi_2(x) = -\frac{x\ (x-R)}{4h_3}\left(\big(4f_2-(10-\varepsilon)h_3\big)\,x + \frac{\varepsilon\, b_1(4f_2-(2-\varepsilon)h_3)}{2+\varepsilon}\right) - \Phi_2'(x),$$

where

$$R = -\frac{b_1\,(4f_2-(2-\varepsilon)h_3)}{2\ (2+\varepsilon)h_3} \qquad \text{and} \qquad r_1 - \frac{2b_1\,(4f_2-(2-\varepsilon)h_3)}{(2+\varepsilon)\,(4f_2+(2-\varepsilon)h_3)}\ .$$

The highest common factor between $\Phi_1$ and $\Phi_2$ is the polynomial $\Phi(x) = x(x-R)$, thereby, lemma 2.3.2 assures that $u_0$ fulfils

$$D(\Phi\, u_0) + \Psi\, u_0 = 0$$

where

$$\begin{aligned}\Psi(x) &= \frac{-1}{4(2+\varepsilon)h_3}\Big((2+\varepsilon)(4f_2-(10-\varepsilon)h_3)\,x + \varepsilon\, b_1(4f_2-(2-\varepsilon)h_3)\Big) - \Phi'(x)\\ &= -\frac{\big(4f_2-(2-\varepsilon)h_3\big)}{4h_3}\ (x+b_1)\ .\end{aligned}$$

Consequently, $u_0$ is a semiclassical form of class zero, ergo $u_0$ is a classical form. More precisely, since $\Phi$ is a second degree polynomial with two distinct roots, $u_0$ is a Jacobi classical form and the sequence $\{P_n\}_{n\in\mathbb{N}}$ represents a Jacobi polynomial sequence. Any affine transformation leaves invariant the classical character, so, in order to determine the range for the parameters associated with a Jacobi form, we consider $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$ defined through $\widetilde{P}_n = \left(\frac{R}{2}\right)^{-n} P_n(\frac{R}{2}x+\frac{R}{2})$, and its associated Jacobi form is given by $\widetilde{u}_0 = \Big(h_{a^{-1}} \circ \tau_{-a}\Big)u_0$, with $a = R/2$. In this case $\widetilde{u}_0$ fulfils the differential functional equation

$$D(\widetilde{\phi}(x)\ \widetilde{u}_0) + \widetilde{\psi}(x)\ \widetilde{u}_0 = 0$$

with

$$\widetilde{\phi}(x) = x^2 - 1 \qquad \text{and} \qquad \widetilde{\psi}(x) = x\left(-\frac{\varepsilon}{4} - \frac{f_2}{h_3} + \frac{1}{2}\right) - \frac{f_2}{h_3} + \frac{5}{2} + \frac{3\varepsilon}{4}$$

So, $\widetilde{u}_0$ represents the (canonical) *Jacobi* form whose parameters $(\alpha,\beta)$ may be obtained by the comparison of the polynomial $\Psi$ obtained here with the expression for the corresponding polynomial displayed in Table 2.1, and we derive that

$$\Big(\alpha,\beta\Big) = \left(\ \frac{\varepsilon}{2}\ ,\ \frac{4f_2-(10+\varepsilon)h_3}{4h_3}\ \right)\ .$$

Additionally, according to the information displayed in Table 2.1, the recurrence coefficients, associated to the second order recurrence relation satisfied by the elements of $\{\widetilde{P}_n\}_{n\in\mathbb{N}}$, are

$$\begin{aligned}
\widetilde{\beta}_n &= \frac{\beta_n - \frac{R}{2}}{\frac{R}{2}} \\
&= -\frac{(4f_2-(10-\varepsilon)h_3)\,(4f_2-(3\varepsilon+10)h_3)}{(4f_2+(8n+\varepsilon-10)h_3)\,(4f_2+(8n+\varepsilon-2)h_3)}\;, \quad n\in\mathbb{N}, \\
\widetilde{\gamma}_{n+1} &= \frac{\gamma_{n+1}}{(R/2)^2} \\
&= \frac{32(n+1)(2n+2+\varepsilon)h_3^2\,(4f_2+(4n-\varepsilon-6)h_3)\,(4f_2+(4n+\varepsilon-6)h_3)}{(4f_2+(8n+\varepsilon-6)h_3)\,(4f_2+(8n+\varepsilon-2)h_3)^{\,2}\,(4f_2+(8n+2+\varepsilon)h_3)}\;, n\in\mathbb{N}.
\end{aligned}$$

Naturally we may consider $\varepsilon = 2\alpha$ and $\frac{f_2}{h_3} = \beta + \frac{5+\alpha}{2}$, and this leads to

$$\widetilde{\beta}_n = \frac{\beta_n - \frac{R}{2}}{\frac{R}{2}} = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\;, \quad n\in\mathbb{N},$$

$$\widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{(R/2)^2} = \frac{4(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2\,(2n+\alpha+\beta+3)}\;, \quad n\in\mathbb{N}.$$

Finally, by virtue of (4.2.8) we obtain a relation between the form $u_0^{[1]}$ and $u_0$. After the conclusions obtained in this case we have

$$4\,x\,u_0^{[1]} = h_3\,x\left(x + \frac{b_1\,(4f_2-(2-\varepsilon)h_3)}{2(2+\varepsilon)h_3}\right)^2 u_0$$

which may be divided by $x$ and we get

$$\begin{aligned}
4\,u_0^{[1]} - 4\,\delta_0 = h_3&\left(x + \frac{b_1\,(4f_2-(2-\varepsilon)h_3)}{2(2+\varepsilon)h_3}\right)^2 u_0 \\
&- \left\langle u_0, h_3x^2 + \frac{b_1\,(4f_2+(\varepsilon-2)h_3)\,x}{2+\varepsilon} + \frac{b_1^2\,(4f_2+(\varepsilon-2)h_3)^{\,2}}{4(2+\varepsilon)^2h_3}\right\rangle\,\delta_0
\end{aligned}$$

Since $\langle u_0, x\rangle = -b_1$, $\langle u_0, x^2\rangle = b_1^2+\gamma_1$ and $\gamma_1 = \dfrac{2\,b_1^2(4f_2-(6+\varepsilon)h_3)}{(2+\varepsilon)(4f_2+(2+\varepsilon)h_3)}$, then by virtue of (4.2.100), we conclude that

$$-\left(h_3(b_1^2+\gamma_1) + \frac{-\,b_1^2\,(4f_2+(\varepsilon-2)h_3)}{2+\varepsilon} + \frac{b_1^2\,(4f_2+(\varepsilon-2)h_3)^{\,2}}{4(2+\varepsilon)^2h_3}\right) + 4 = 0$$

and consequently we obtain

$$u_0^{[1]} = \frac{h_3}{4}\,(x-R)^2\,u_0 \tag{4.2.123}$$

with

$$R = -\frac{b_1\,(4f_2-(2-\varepsilon)h_3)}{2\,(2+\varepsilon)h_3}$$

The relation (4.2.123) is indeed the key to derive all the needed information about the sequence $\{P_n^{[1]}\}_{n\in\mathbb{N}}$, apart from knowing its recurrence coefficients. In order to accomplish so, we multiply on the left the relation (4.2.123) by the polynomial $f(\cdot)$ which is given by (4.2.101) and we get

$$f(x)\, u_0^{[1]} = \frac{h_3}{4}\, (x-R)^2\, f(x)\, u_0\ .$$

By virtue of (4.2.7) the right hand side of the previous equality may be written just in terms of $u_0^{[1]}$ and $(u_0^{[1]})'$, precisely we have

$$h_3 x\, (x-R)^2\, (u_0^{[1]})' + \left\{ \frac{2-\varepsilon}{4}\, h_3 (x-R)^2 - f(x) \right\} u_0^{[1]} = 0$$

which may be transformed into

$$\Big( h_3\, x\, (x-R)^2\, u_0^{[1]} \Big)' + \Big( -\big(h_3\, x\, (x-R)^2\big)' + \frac{2-\varepsilon}{4}\, h_3\, (x-R)^2 - f(x) \Big)\ u_0^{[1]} = 0\ .$$

Recalling the obtained expression for the polynomial $f(\cdot)$ given in (4.2.101), we thus have that $u_0^{[1]}$ is a semiclassical form of class lower or equal to $1$ fulfilling the equation

$$\Big( \widehat{\Phi}(x)\, u_0^{[1]} \Big) + \widehat{\Psi}(x)\, u_0^{[1]} = 0$$

with

$$\begin{aligned} &\widehat{\Phi}(x) = x\, (x-R)^2 \\ &\widehat{\Psi}(x) = -\widehat{\Phi}'(x) + (x-R) \left\{ \frac{2-\varepsilon}{4}(x-R) - \frac{1}{h_3} \left( f_2\, x + \frac{b_1\big(4f_2 - (2-\varepsilon)h_3\big)}{8} \right) \right\}\ . \end{aligned}$$

The polynomial $\widehat{\Psi}(\cdot)$ may be rewritten as follows

$$\widehat{\Psi}(x) = -\widehat{\Phi}'(x) + (x-R) \left\{ \left( \frac{2-\varepsilon}{4} - \frac{f_2}{h_3} \right) x + \frac{\varepsilon}{2} \right\}$$

so, we easily observe that

$$\left(\vartheta_R^2 \widehat{\Phi}\right)(x) + \left(\vartheta_R \widehat{\Psi}\right)(x) = -\left( \frac{6+\varepsilon}{4} + \frac{f_2}{h_3} \right) x + \frac{2+\varepsilon}{2}\, R$$

providing

$$\left\langle u_0^{[1]}, \left(\vartheta_R^2 \widehat{\Phi}\right)(x) + \left(\vartheta_R \widehat{\Psi}\right)(x) \right\rangle = \left( \frac{6+\varepsilon}{4} + \frac{f_2}{h_3} \right) b_1^{[1]} + \frac{2+\varepsilon}{2}\, R$$

Since $b_1^{[1]} = \frac{2+\varepsilon}{2(4+\varepsilon)}\, b_2$, then recalling (4.2.99) and the expression for the root $R$ we conclude

$$\left\langle u_0^{[1]}, \left(\vartheta_R^2 \widehat{\Phi}\right)(x) + \left(\vartheta_R \widehat{\Psi}\right)(x) \right\rangle = 0$$

As a result $u_0^{[1]}$ is a classical form fulfilling

$$\Big(x(x-R)\,u_0^{[1]}\Big) + \left\{-\left(\frac{6+\varepsilon}{4} + \frac{f_2}{h_3}\right)x + \frac{2+\varepsilon}{2}\,R\right\}\,u_0^{[1]} = 0\ .$$

The fact that $R \neq 0$ provides $u_0^{[1]}$ to be a Jacobi form. In order to determine the parameters, we consider the equation fulfilled by the form $\widetilde{v}_0 = (h_{a^{-1}} \circ \tau_{-a})\ u_0^{[1]}$ with $a = R/2$, which is

$$\Big((x^2-1)\,\widetilde{v}_0\Big) + \left\{-\left(\frac{6+\varepsilon}{4} + \frac{f_2}{h_3}\right)x + \frac{2+\varepsilon}{2}\,R\right\}\,\widetilde{v}_0 = 0$$

Following the information displayed in Table 2.1, $\widetilde{v}_0$, as well as $u_0^{[1]}$, is a Jacobi form of parameters $(\widehat{\alpha}, \widehat{\beta})$ with

$$\widehat{\alpha} = \frac{\varepsilon}{2} \quad \text{and} \quad \widehat{\beta} = -\frac{1}{2} - \frac{\varepsilon}{4} + \frac{f_2}{h_3}$$

To sum up, under the assumption $\deg h = 3$, $\{P_n\}_{n\in\mathbb{N}}$ is, up to a linear change of variable, a Jacobi MOPS of parameters $(\alpha, \beta)$, while $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is a Jacobi MOPS with parameters $(\alpha, \beta+2)$ .

### 4.2.4 Some comments on the $\mathcal{F}_\varepsilon$-classical sequences

The $\mathcal{F}_\varepsilon$-classical sequences are indeed Laguerre sequences or Jacobi sequences, whether $\deg h = 1$ or $\deg h = 3$. There is no possibility of having $\deg h = 0$ or $\deg h = 2$.

Perhaps the most interesting dichotomy to be considered here lies in the Appell character. Precisely, if the $\mathcal{F}_\varepsilon$-classical sequences are Appell sequences, then they must be a Laguerre sequence of parameter $\varepsilon/2$, otherwise they are Jacobi sequences of parameters $\left(\frac{\varepsilon}{2}, \mu - \frac{\varepsilon}{4}\right)$ with $\varepsilon \neq -2(n+1)$, $n \in \mathbb{N}$, $4\mu - \varepsilon \neq -4(n+1)$ and $4\mu + \varepsilon \neq -4n$ for all the nonnegative integers $n$. Moreover, if the MOPS $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{F}_\varepsilon$-classical but not possessing the Appell character, then it is a Jacobi form of parameters $\left(\frac{\varepsilon}{2}, \mu - \frac{\varepsilon}{4}\right)$ and the sequence of $\mathcal{F}_\varepsilon$-derivatives here denoted as $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is also a Jacobi sequence of parameters $\left(\frac{\varepsilon}{2}, \mu - \frac{\varepsilon}{4} + 2\right)$ .

# Postlude

In this work, we have considered the quadratic decomposition of Appell sequences with respect to the lowering operators $D$ and $\mathcal{F}_\varepsilon$. The associated sequences obtained by this approach are still Appell sequences with respect to the lowering operators $\mathcal{F}_\varepsilon$ and $\mathcal{G}_{\varepsilon,\mu}$, respectively. We have indeed plunged into a more general problem, by proceeding to the quadratic decomposition of a $\mathcal{L}_k$-Appell sequence, where $\mathcal{L}_k$ denotes a lowering (differential) operator consisting of the product of the derivative operator by a polynomial with constant coefficients in the powers of $xD$: $\mathcal{L}_k := D\, f(xD)$ where $f(\cdot)$ is a polynomial of degree $(k-1)$ (for $k \in \mathbb{N}^*$) with complex coefficients. Based on *Faa di Bruno's formula* we have

$$D^n\big(p(\zeta^2)\big)(x) = \sum_{\nu=0}^{\lfloor n/2 \rfloor} \frac{2^{n-2\nu}\, n!}{(n-2\nu)!\; \nu!}\, x^{n-2\nu} \left(D^{n-\nu} f(x^2)\right), \quad n \in \mathbb{N},$$

with $\lfloor z \rfloor$ denoting the integer part of the number $z$, which enables us to proceed to the quadratic decomposition of the $\mathcal{L}_k$-Appell sequences in an analogous manner as the one adopted in sections 3.1 and 3.4. After a number of computations, we reach the conclusion that the four associated sequences obtained by this approach are Appell sequences with respect to a new differential operator, $\widetilde{\mathcal{L}}_{2^k} = D\, g_\varepsilon(xD)$, where $g_\varepsilon(\cdot)$ is a polynomial which depends on the parameter $\varepsilon$ that is either 1 or -1, and such that $\deg g_\varepsilon = (2^k - 1)$ with $k \in \mathbb{N}^*$. However, this problem has revealed to be too widespread and, as we do not envisage a larger significance of this, we have restricted ourselves, with the concomitant implication in a lack of details, to these final comments.

As we stated in section 3.8, the four sequences resulted from the quadratic decomposition of a $q$-Appell sequence are also Appell sequences but with respect to a new operator denoted therein as $\mathcal{M}_q$. Such $\mathcal{M}_q$-Appell sequences need further study. As matter of fact, the goal of this section is to trigger the attention to many other good results that may be obtained through this approach. We have definitely put in practice the quadratic decomposition of other Appell sequences with respect to other lowering operators. In some cases, it appears to be much more natural to consider a more general quadratic decomposition - see the work of Macedo [77], however, we limited ourselves to settle on the most simple and illustrative examples. Another associated and also ongoing problem is concerned with the quadratic decomposition of the *Dunkl-Appell* sequences. Since the work of Ghressi and Khérigi [49] it is already known that if a symmetric polynomial sequence is both Dunkl-Appell and orthogonal then it is, up to a linear change of variable, the *generalised Hermite polynomials* $\{\widetilde{H}_n^{(\mu)}\}_{n\in\mathbb{N}}$ widely studied by Chihara [26]. The quadratic decomposition of these last is well known and is given by

$$\widetilde{H}_{2n}^{(\mu)}(x) = L_n^{(\mu-\frac{1}{2})}(x^2) \quad ; \quad \widetilde{H}_{2n+1}^{(\mu)}(x) = x\, L_n^{(\mu+\frac{1}{2})}(x^2)\ ,\ \mu \neq -n - \frac{1}{2},\ n \in \mathbb{N},$$

where $\{L_n^{(\alpha)}\}_{n\in\mathbb{N}}$ represent the Laguerre polynomials. The similarities with the other problems considered in chapter 3 are quite evident.

To sum up, the quadratic decomposition of Appell sequences with respect to lowering operators appears to be a very powerful tool when combined with the study of the corresponding classical sequences. It is also undoubtedly true that it is often tough to deal with Hahn's problem generalised to lowering operators. After the analysis carried out in Chapter 4, we realise how thorny this problem may become. At last but not least, within this framework there still is a considerable amount of material to be explored.

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# Index

### Symbols



### A



### B